Springer Monographs in Mathematics

Jean-Pierre Bourguignon

Variational Calculus

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LA VISION[1]

D'où est sorti ce livre

Il n'est pas de brouillards, comme il n'est point d'algèbres,
Qui résistent, au fond des nombres ou des cieux,
A la fixité calme et profonde des yeux ;
Je regardais ce mur d'abord confus et vague,
Où la forme semblait flotter comme une vague,
Où tout semblait vapeur, vertige, illusion ;
Et, sous mon œil pensif, l'étrange vision
Devenait moins brumeuse et plus claire, à mesure
Que ma prunelle était moins troublée et plus sûre.

Victor HUGO
in *La Légende des Siècles*

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[1] This is an excerpt from the introduction of the famous (long!) collection of poems by Victor Hugo called "*La légende des siècles*", created in the period 1859–1883, aiming at presenting his poetary vision of the history and evolution of humanity (note that the title and subtitle of the excerpt are due to him):

THE VISION

From where this book comes

There is no fog, as there is no algebra,
Which persist, in the depths of numbers or of the skies,
To the quiet and deep gaze of the eyes;
I looked at this wall, at first confused and vague,
Where form seemed to float like a wave,
Where all seemed to be mist, vertigo, illusion;
And, under my pensive eye, the strangest vision
Became less foggy and clearer,
As my eye was less troubled and surer.

Editor's Preface

As pointed out in the preface to this book, the Calculus of Variations underlies many of the applications of mathematics to the natural sciences. In particular, the *Euler–Lagrange equations* are the universal form of the specific equations that govern essentially all the entities of fundamental physics, from the elementary constituents of matter to the forces that hold them together to the structure of space and time. They are also the starting point for the most natural approach to the quantum theory of fields. It is a monumental achievement of humanity to have discovered them in the concise and elegant form that they now have. The readers of this book will encounter their theory and use at the hands of a master and thereby come to appreciate their profundity for themselves.

Jean-Pierre Bourguignon has become widely known and respected in the global mathematical community as a great leader on account of the distinguished service roles he has most generously assumed over the years. These include his directorship of the IHES, one of the truly great research institutions in the world, and his presidency of the European Research Council. His prominence in the socio-politics of mathematics may lead some to momentarily forget his outstanding contributions as a researcher and his remarkable skills as a teacher. He is a world-leading expert on the geometric variational equations of mathematical physics, while several generations of excellent students at the École Polytechnique have been introduced to the profound and beautiful world of global analysis by way of his inspiring lectures.

As is well-known, the Springer Monographs have as their primary goal the publication of research monographs dealing with rather recent advances. However, every now and then, a pedagogical work of great importance and expositional excellence is put forward because of its role as an enabler of current and future research at the highest level. As a specimen of this kind of publication, the book in your hands will have few equals. It is with great pleasure that we offer it within this series.

Edinburgh and Seoul, *Minhyong Kim*
September 2022

Preface

The Calculus of Variations aims to characterise a mathematical object such as a curve, a surface, a function, or more generally a field (in the physicists' sense), which satisfies "*variational*" conditions. In more modern language, this means that this object should be an *extremum of a function* (in this context, one often speaks of *functionals*) defined on the space of objects which are in competition.

Problems arising from the Calculus of Variations have played an important role in the history of mathematics. Substantial contributions to problems of this type were made by, among others, Bernoulli, Euler, Lagrange, Leibniz, Jacobi, Hamilton, Weierstrass, Hilbert, Carathéodory, Morse. This subject also attracts the attention of numerous present-day mathematicians and, thanks to a considerable extension of the methods available, important new developments have appeared recently with spectacular applications, both inside and outside mathematics.

The Calculus of Variations has maintained, and still maintains, a close relationship both with analysis and with geometry. The progress which it inspired has touched the foundations of mathematics as well as the elaboration of new methods. Today, it remains a source of problems of great interest and nourishes a very active branch of the theory of partial differential equations. Many classical variational problems in geometry have recently been the object of intense research activity in theoretical physics (we shall mention them on several occasions in this book).

This course aims to present the basic concepts which allow one to discuss several classical problems of the Calculus of Variations. As well as giving general methods, it also studies several special situations. It is centred around the *search for the extrema of a function defined on a space*. To accomplish this, it is necessary to generalise the notion of a space in two directions: first, to treat effectively the objects which are "varied" (usually functions), one is naturally led to work with *infinite-dimensional spaces* (and there lies the beginning of functional analysis, which has proved to be so powerful for the solution of partial differential equations); second, to find the extrema of the function being studied, one must have available a notion of the derivative on spaces of curves, which are the *configuration spaces* featuring most often in concrete situations, such as in mechanics. It is there that we shall encounter *intrinsic differential geometry* for the first time; this part is often called *differential calculus*. We have deliberately used geometric language because it seems to us the most suitable and most effective for treating the problems we have in mind, whence the title *Variational Calculus* of this book.

Another idea which appears here repeatedly is this: relate the *global* form of a space to the properties of the functions defined on it, which physicists like to call *observables* (a terminology which we have retained). The interest which this approach excites today, both inside and outside mathematics, has led to the introduction of a special term to denote it. One speaks of "*Global Analysis*", or in German "*Variationsrechnung im Großen*" (the old terminology in English was "*Calculus of*

Variations in the Large"). For this reason, we were tempted to add to the title Variational Calculus the adjective *global.* We have decided against this, however, since in the limited space we have available, it is difficult to go very far in this direction. In fact, a more profound study would necessitate excursions into more advanced areas of mathematics such as algebraic topology and differential topology. Moreover, the most spectacular applications make extensive use of results from the theory of partial differential equations.

In many places another theme appears: the study of the *symmetries* of a system, which Felix Klein, in his Erlangen programme of 1872, showed is identical to the search for properties which are invariant under the action of a group. These "*continuous groups of transformations*" (the action of which is the mathematical formulation of symmetry) have taken up, thanks to the work of Sophus Lie, a privileged place in the Calculus of Variations. Moreover, the presence of symmetries in a variational problem is accompanied, according to a theorem of Emmy Noether, by that of *conserved quantities*, which often facilitate the solution of a problem.

The climax of the book occurs in the chapters devoted to the *Euler–Lagrange equations* and *Hamilton's equations.* These equations are very far-reaching (one is seriously tempted to say "*universal*"), and underlie numerous theories in diverse branches of physics, mechanics and economics. The multiplicity of situations where they are used successfully testifies to the power of the concepts elaborated by mathematicians for solving these problems. This shows how completely wrong was Felix Klein's judgement on this subject, as expressed in his presentation of the development of mathematics in the XIX$^{\text{th}}$ century. "*Trotz der unzweifelhaften Schönheit dieses Gebietes möchte ich jedoch vor einem einseitigen Studium warnen..... In der Tat kann der Physiker wenig, der Ingenieur gar nichts von diesen Theorien für seinen Aufgaben brauchen. Sie sind sozusagen ein Schema mit leeren Fächern, in welche die bunte Welt der Erscheinungen erst eingeordnet werden muß, um sie sinnvoll erscheinen zu lassen*".[2] In the edition of these lectures edited in 1926 by Richard Courant and Otto Eduard Neugebauer under the title "*Vorlesungen über die Entwicklung der Mathematik im 19. Jahrhundert*" (volume XXIV of the Springer series *Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen*), they inserted the following footnote: "*Wir haben diese, durch die Entwicklung der letzten Jahren widerlegt Bemerkungen stehen lassen, da sie zu der gerade von Klein oft gennanten Erscheinung einen Betrag liefern, wie scheinbar rein mathematische Theorien auch für die Nachbarwissenschaften unvermutet von größter Bedeutung werden.*"[3] This historical reference, in which the actors are among the most eminent mathematicians, emphasises *the great difficulty involved in judging* a priori *the applicability of a mathematical result.*

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[2] "*Despite the undoubted beauty of this field, however, I would caution the reader.... In reality, physicists can extract only very little from these theories for use in their own work, and engineers almost nothing. They constitute a kind of scheme with empty spaces, which only takes on meaning when the gaps are filled from the varied world of natural phenomena.*"

[3] "*We have retained these remarks, which are refuted by recent developments, because they are an example of a phenomenon often evoked by Klein himself, namely, that mathematical theories which may seem completely pure can turn out to be extremely relevant to other sciences.*"

Today, the Euler–Lagrange equations form the conceptual framework for all studies, based on energy methods, of the *mechanics of systems defined by a finite number of parameters*. The dual methods related to Hamilton's equations are now used in practice in numerous questions of *automation*, especially in problems arising from *spatial engineering* and from *econometrics*. They also serve as the point of departure for *quantum mechanics*, which cannot be developed starting from the Newtonian view of mechanics, and as the setting for the theory of fields in physics. They illustrate the power of the notion of *duality*, once it has been suitably extended to this setting. Note, however, that the practical study of these questions often cannot be divorced from that of the numerical approximation of their solutions. These considerations lead to the introduction of the idea, which can sometimes be forgotten, that *in real life (and in industry), there are problems of a mathematical nature whose solution requires the use of among the most recent mathematical results, and which give rise to new research which can be carried out only by teams of mathematicians developing the most modern theories.* In this process, the modelling of the phenomena which one is studying plays, of course, an essential role.

In a book of this level, it is difficult to include the newest results. We have, above all, tried to put the recent evolution of the subject in perspective, and to bring out a truth which is unfortunately often concealed: "*mathematics is, today more than ever, a living science open to interaction with other sciences*". In the last forty years, its development has undergone an unprecedented acceleration, with the result that there are probably more professional mathematicians alive today than have existed before since the birth of humanity. Such an explosion makes the selection of basic material more difficult than ever, since this expanding activity goes hand-in-hand with a very rapid diversification accompanied by a multitude of the most unexpected cross-fertilisations between sub-disciplines of mathematics.

The prerequisites for this book are those acquired by most students in the first two years of an undergraduate course. We regularly appeal to the fundamental results of linear algebra. An appendix collects together the basic ideas and concepts of general topology.

The subtlety of the knowledge contained in this book may not be as surprising to the reader as the *change of style* which it exhibits compared to previous courses. This change is, in part, inherent in the fact that this is a more advanced course; but one can find another explanation in the general evolution of styles of exposition which reflects the manner in which knowledge is developing: to a period of linear development of mathematics centred around the structure of great theories corresponds a "one-dimensional" style of presentation; to the recent explosion of sub-disciplines and multiple interactions corresponds a more personal synthesis. From these considerations follows, in particular, the great importance of the bibliography, since a book on mathematics offers, as well as the presentation (correct, if possible!) of theorems, a vision of the connections between concepts. In the opinion of this author, a real understanding of the fundamental ideas can only develop out of the diversity of these approaches.

Palaiseau, 4 August 1987 *Jean-Pierre Bourguignon*

Preface to the English Edition

This book is the English translation of the notes of a course delivered more than 30 years ago, several years in a row, to the entire student cohort at École Polytechnique (more than 400 students each year). Because of the special context of the teaching at the school, the level of the course is between undergraduate and graduate. Indeed, students there are selected on the basis of their rather high level of competence in mathematics. They are exposed to lectures in different sciences and also some in the social sciences and humanities. The lectures were delivered in a large lecture hall on the basis of the notes. They were complemented by exercise sessions, hence the inclusion of a number of exercises in the notes that could be used during these sessions – without being an obligation since some teachers preferred to introduce their own.

Because of the time passed, a number of adjustments of dates mentioned in the text were necessary and have been made. This edition also provided an opportunity to be more systematic about the introduction of the short bibliographical notes of scientists whose names appear in the text.

I am very grateful to Professor Andrew Pressley, who did a very careful and faithful job translating the notes. He completed his work 30 years ago and I thank him also for his patience. We owe the existence of this edition to the persistence of the mathematical editors at Springer, now Springer Nature. They managed to lead this project to its fruition as a volume in the ‘Springer Monographs in Mathematics’ series. I am thankful to them for that. I also owe a lot to the students and my colleagues at École Polytechnique for improving the text and spotting typos and places where it needed to be clarified, if not plainly corrected. Any inaccuracies that remain are my sole responsibility.

I dedicate this book to all my mathematics teachers Mr Lemaître, Mr Thovert, Mr Martin, Mr Gontard, Mr Choquet and Mr Berger, who exposed me to different ways of looking at mathematics and who, unfortunately, have all left us.

Bures-sur-Yvette, *Jean-Pierre Bourguignon*
21 July 2022

To the Reader

These lecture notes comprise 11 chapters which are *naturally decomposed into three parts*, each being better approached with a different mindset.

The first, entitled "*The Analytic Setting*", covers Chapters I, II and III. Its purpose is to amplify the students' knowledge of analysis. It is a kind of warm-up.

The second, entitled "*The Geometric Setting*", covers Chapters IV, V, VI and VII. It introduces another approach, involving new concepts. It requires some reflection, and the reader must practice a number of exercises to become comfortable with the new language.

The third, entitled "*The Calculus of Variations*", covers Chapters VIII, IX, X and XI. This is the real goal of the course. It contains many applications coming from different fields, and it is this variety that gives the presented theorems their value. One must be perseverant enough to embrace all facets of the theory. In this part, solving a number of exercises drawn from a wide range of different topics is compulsory.

Some typographical remarks about the text: Chapters are numbered using Roman numerals while references to the appendix use Arab numerals. Within each chapter, the numbering is strictly linear to facilitate internal cross-references; sections are ordered alphabetically to visually organise the text... and the table of contents. At the beginning of each section, one finds a short résumé of its content.

We use two sizes of characters: the *normal size* is that of the largest part of the text; a *smaller size* is used for some complementary material for readers interested in extensions of the notions presented. We also use the latter for proofs of theorems, propositions and lemmas. This distinction has two advantages: it gives some rhythm to the text, avoiding an excessively uniform appearance. It also suggests something more substantial, namely that, during a first reading, proofs can be skipped.[4]

The text contains a number of *exercises*. One of the objectives is to remind readers that it falls on them to control their understanding while progressing in their study.

[4] This gives me an opportunity to recall that a mathematical text should ideally be read three times: the *first* time reading only definitions and statements, even if one gets the feeling of having lost track at some stage; the *second* time forming a precise idea about each statement, still avoiding reading proofs; and the *third* one coming finally to a complete and thorough reading of the whole text.

A warning though: the exercises are of a variable difficulty, but the reward of having tried seriously to solve an exercise is almost as great as that of resolving it. Exercises appearing in smaller characters are likely to be more difficult and are reserved for motivated readers.

Each chapter ends with short *historical notes*. These are rather incentives to learn more than complete stories. When the name of a scientist appears in the text, a short *bibliographical reference* is provided as a footnote. The objective is in particular to remind readers that *mathematics is a living science produced by human beings* in which the birth of a new concept takes efforts by many who, sometimes, can get bogged down in dead-ends.

At the end of the book, indexes are provided, one for *notation* and the other for *terminology*.

Last but not least, a short *bibliography*, deliberately selective and multilingual, aims at encouraging readers to explore other viewpoints on the theme of this course and also to discover some further developments.

Bures-sur-Yvette, *Jean-Pierre Bourguignon*
21 July 2022

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