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Excursions in Multiplicative Number Theory

With contributions by Pieter Moree and Alisa Sedunova

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Preface

This book is concerned with the arithmetic of integers, i.e. with the behaviour of (usually positive) integers with respect to addition and multiplication. One major way to describe this is through the study of *arithmetical functions*. Such functions often behave in a locally erratic but globally predictable way. For example, their mean value up to n is generally close to some simple function of n. It is this feature of global regularity that allows one to study their behaviour using methods from analytic number theory.

The goal of this book is to teach the readers how *to establish* such results. Since we encourage an active attitude with a focus on methods rather than performance, we have included nearly 300 exercises that guide the readers towards proving results. These in themselves may not be the best available, but some exercises in the later chapters ask the readers to prove results that are close to the forefront of current research.

As mentioned above, the typical behaviour of an arithmetical function is irregular, especially when its definition depends on the factorization structure of n. Consider for instance the function

$$f_0(n) = \prod_{p|n}(p-2). \tag{0.1}$$

The introduction of some notation is called for. Here, and in the whole book, the letter p (or p_1, p_2, etc.) stands for a prime number, while the notation "$p|n$" means that we are considering the set of primes p that divide n. For $n = 1, 2, \ldots, 54$, we find the following values of $f_0(n)$:

$$1, 0, 1, 0, 3, 0, 5, 0, 1, 0, 9, 0, 11, 0, 3, 0, 15, 0, 17, 0, 5, 0, 21, 0, 3, 0, 1, 0, 27,$$
$$0, 29, 0, 9, 0, 15, 0, 35, 0, 11, 0, 39, 0, 41, 0, 3, 0, 45, 0, 5, 0, 15, 0, 51, 0$$

.

This list does not give us much information (other than $f_0(n)$ vanishes when n is even), but we shall show that its *average* (*mean value*) behaves in a regular way.

The book in itself is split in five parts that we now describe briefly.

Approach.

Before embarking on computing average orders, we spend some time exploring arithmetic functions and notions of *arithmetical* interest. Multiplicativity is a fundamental notion there, and we develop a calculus on multiplicative functions that renders their handling very easy. The basics on Dirichlet series is then introduced and rough estimates concerning the growth of multiplicative functions are proved. Additional chapters on the Legendre symbol and its Dirichlet series; formulas akin to the Möbius inversion formula (some of which are new!) conclude the introductory part.

The Convolution Walk.

The first guided walk to higher ground exposes the readers to "elementary" methods, the focus of which is to prove the next theorem.

Theorem $\mathscr{A}$

For all real positive x we have

$$\frac{1}{x}\sum_{n\le x} f_0(n) = \tfrac{1}{2}\mathscr{C}_0 x + O^*(3.5\,x^{3/5}),$$

where $\mathscr{C}_0$ is a constant given by

$$\mathscr{C}_0 = \prod_{p\ge 2}\left(1-\frac{3}{p(p+1)}\right) = 0.29261\,98570\,45154\,91401\cdots \tag{0.2}$$

Here and in the sequel, we write $f = \mathcal{O}(g)$ as a shorthand for there being a positive constant C such that $|f| \le Cg$. This is the Landau big-$\mathcal{O}$ notation that is standard in mathematics. We supplement it with the notation $\mathcal{O}^*$; we say $f = \mathcal{O}^*(g)$ if $|f| \le g$. This notation is very practical when one wants to compute explicit bounds for the intermediary error terms.

The above theorem shows that, by taking into account many values of f_0, the influence of the aberrant ones is swept under the rug and a simple regularity is brought to the fore. This theorem is proved by comparing the function f_0 to a simpler one (here $n \mapsto n$) that is easier to analyse.

This first walk is based on the relatively simple nature of f_0. The convolution method is folklore (though only a few systematic expositions are available, e.g. [2]). It is very flexible and allows one to obtain excellent error terms. We present several examples on how to use it. A similar philosophy is at work when one wants to compute *Euler products* (such as the product in (0.2)) and *Euler sums* and we dedicate a full chapter to this issue. We conclude this part with the Dirichlet hyperbola formula, as it is the second tool at our disposal to compute average orders elementarily.

The Levin–Faĭnleĭb Walk.

The general idea of this second walk is to *deduce* the evaluation of mean values of multiplicative functions from the behaviour of our function on prime powers. We give in particular a completely explicit version of the Levin–Faĭnleĭb Theorem. Here is the consequence that we have in store.

Theorem $\mathscr{B}$

When $x \geq \exp(20\,000)$ and $d(n)$ is the number of divisors of n, we have

$$\sum_{n \leq x} \frac{\sqrt{d(n)}}{n} = \mathscr{C}_1 (\log x)^{\sqrt{2}} \big(1 + O^*(20000/\log x)\big)$$

where

$$\mathscr{C}_1 = \frac{1}{\Gamma(1+\sqrt{2})} \prod_{p \geq 2} \left\{ \left(1 + \sum_{\nu \geq 1} \frac{\sqrt{\nu+1}}{p^\nu}\right)\left(1 - \frac{1}{p}\right)^{\sqrt{2}} \right\}. \tag{0.3}$$

Getting an accurate value of $\mathscr{C}_1$ is a challenge, we leave to the readers. In effect, this method transfers regularity of the function on the primes to regularity on the integers. But to be able to apply it, we need to detect the regularity on the primes! We devote two chapters to this question and prove several classical estimates of the sort concerning logarithmic averages. For instance, $\sum_{p \leq x} (\log p)/p$ is shown to be asymptotic to $\log x$. We also prove a similar estimate when p is restricted to a congruence class modulo 3 or 4, but we are not able at this level to dispense of the *logarithmic* average and to prove, for instance, the Prime Number Theorem, i.e. that $\sum_{p \leq x} \log p$ is asymptotic to x. We are thus unable to provide an asymptotic for $\sum_{n \leq x} \sqrt{d(n)}$. We however have enough material to compute some Euler products and sums where the prime is restricted to a congruence class modulo 3 or 4; a chapter is dedicated to this task. We conclude this part with an application of our accumulated expertise and prove an asymptotic for $\sum_{n \leq x} d(n^2+1)$.

The Mellin Walk.

The third walk goes through analytical landscape where the information on our function is taken from its *Dirichlet series*. One feature is that we introduce a regular weight function $F(t)$ and consider expressions of the form $\sum_{n \geq 1} f_0(n) F(n/x)$. As the sum now involves every natural integers, it captures some aspect of the global behaviour of f_0, provided, of course, that $F(t)$ is sufficiently smooth and tends quickly enough to zero as t tends to infinity in order for the sum to represent a well-defined function of x. Theorem $\mathscr{C}$ is a typical example of a result involving a weight function.

Theorem $\mathscr{C}$

Let x be real and positive. We have

$$\sum_{n\geq 1} f_0(n)e^{-n/x} = \mathscr{C}_0 x^2 + O^*(133 \cdot x^{7/4}),$$

where $\mathscr{C}_0$ is the constant defined in (0.2).

As a matter of fact, one can infer the asymptotic given in Theorem $\mathcal{A}$ from this result, and this is the theme developed in Chap. 22.

Mellin transforms are also a powerful tool to prove the Prime Number Theorem, and we finally achieve this in Chap. 23. We do so in a completely explicit manner in the form of an inequality satisfied by the summatory function of the Möbius function.

Higher Ground.

In the final part, we introduce extensions and applications based on the classical techniques that were gently introduced and commented on in the earlier parts of the book. Non-negative multiplicative functions were our main focus in the first four parts. In the final part, we consider mean values of some oscillating multiplicative functions, and in particular, we give an explicit bound for $\sum_{n\leq x}\mu(n)/\varphi(n)$. We also present P_k-numbers (numbers having at most k, not necessarily distinct, prime factors) and delve deeply enough into the theory of the Brun sieve to prove a *fundamental lemma*, i.e. a wide-ranging but rather sharp upper bound for the quantities considered in sieve, as the number of primes in an interval for instance. This will enable us to evaluate the two exponential sums $\sum_{p\leq x}\exp(2i\pi\rho p)$ and $\sum_{n\leq x}\mu(n)\exp(2i\pi\rho n)$ with modern techniques and to prove, for instance, that the two sequences $(\cos 2\pi\rho p)$, when p ranges, over the primes and $(\cos 2\pi\rho n)$, when n ranges over the integers having an even number of prime factors, are dense in $[-1,1]$. We will end this journey with a short presentation of the large sieve and of a practical arithmetical form of it due to H.L. Montgomery.

An Active Teaching.

Since we aim at keeping the readers on their toes, and since we shall also explicitly bound most of the intermediary error terms, we have decided to give this monograph an algorithmical streak. This will help the readers in carrying out experiments. Computation is a way to get better mastery and insights into the mathematical topic one studies. B. Riemann, one of the greatest mathematicians of all time, had extremely large sheets of paper, on which he performed computations. For example, he computed by hand the location of the first few zeros of the Riemann zeta-function. Study of these sheets has shown that Riemann kept many findings up his sleeve, the formula now known under the name “Riemann-Siegel formula” being an example. Riemann developed it in order to have a faster and more accurate way of computing the Riemann zeta-function in the critical strip. Another brilliant mathematician known for carrying out many numerical calculations was C.F. Gauss. He became famous in 1802 for predicting the position of the planet-like object Ceres. The prediction required Gauss to do an enormous amount of calculation and its purpose was to help astronomers find Ceres again in the sky after it had been too close to the sun’s glare for months to confirm the first observations by Piazzi in 1801.

Pari/GP [6] and Sage [8] are the two computer algebra packages that we shall use, in version 2.11 at the time of writing for Pari/GP and in version 9 (with Python 3) for Sage. They are freewares and developed by a dedicated community that guarantees the accuracy of the results. Numerical precision is always an issue and we do not dwell on it here, but the readers are expected to document themselves on this issue if they want to obtain trustworthy results. Furthermore, in Sage, we have *interval arithmetic* at our disposal: results are expressed as a couple (E^-, E^+) representing an (unknown) real number E in the interval $[E^-, E^+]$. Let us elaborate a bit on this issue here. The script

```
R = RealIntervalField(64)
sqrt(R(2))
sqrt(R(2)).upper()
sqrt(R(2)).lower()
```

answers successively 1.4142135623730950488? with the last digit 8 being maybe wrong: the ?-sign means that it may be 7 or 9, then 1.41421356237309505 followed by 1.41421356237309504 where this time, all the digits are correct: the inequalities

$$1.41421356237309504 \le \sqrt{2} \le 1.41421356237309505$$

are *certified*. A word is surely required on the Sage syntax: `sqrt(R(2))` is an *object* that contain several fields: here we have an upper bound and a lower bound; to have access to these fields, we have *accessory functions* at our disposal, here `upper()` and `lower()` that come as further specification, which explains the syntax `sqrt(R(2)).upper()`. Pari/GP sometimes also uses this syntax.

Neither Pari/GP nor Sage are extremely fast and for specific purposes it is better to use some C-code directly; Pari/GP has the advantage that one may automatically derive some more efficient C-code from most Pari/GP scripts by using the free software `gp2c`. Furthermore, the ease of usage of both packages compensates for the relative loss of speed.

Notation.

We end this introduction by commenting on our notation.

- It is common in multiplicative number theory to write $\sum_{p\geq 2} f(p)$ in the case that the variable p is restricted to *prime* values. The same applies to more intricate expressions and to products as well (cf. our definition (0.1) of f_0). As a rule of thumb, a variable named p is assumed to be prime unless specified otherwise.
- We shall often need to refer to the gcd ("greatest common divisor") of two integers, say a and b. The notation $\gcd(a,b)$ is explicit, but is often shortened to (a,b). The readers may thus find expressions like $f((a,b))$ referring to the value of the function f on the gcd of a and b, or $\sum_{n\leq q,(n,q)=1} 1$ to denote the number of integers below q that are coprime with q (and this number is exactly the value of the Euler φ-function at q).
- Although we try to be very precise, we sometimes prefer to loosen our methodological constraints and use $\mathcal{O}$ rather than $\mathcal{O}^*$. We shall also use the expression $f \ll g$ to mean that $f = \mathcal{O}(g)$. When we add subscripts, like in $f \ll_r g$, it means that there exists a constant C that may depend on r such that $|f| \leq C \cdot g$.

Acknowledgment.

THE BOOK HAS ITS ROOTS in three courses in french given by the author, who with the help of his contributors, Pieter and Alisa, translated the original course material into a more structured write-up. The author did much of the writing and decided the layout of the book, while the contributors added exercises, edited language, put references, wrote some passages and proofread earlier versions. Their work supported the author tremendously; he felt like a frontiersman in uncharted territory, with two companions keeping his back free.

THANKS are due to Kam Hung Yau for his careful checking of this book and for his precious advices.

Further Reading

Different and complementary approaches to arithmetic can be found in several textbooks that can be read without too much prerequisite. Let us mention the classical [1] of T. Apostol, the book [9] of W. Schwarz and J. Spilker dedicated to arithmetical functions, and the more advanced book [5] by H.L. Montgomery and R.C. Vaughan. Other rich sources of information are the books [7] by P. Pollack and [3] by O. Bordelles. A major reference on mathematical computations with Sage is the open access book [4], written and maintained up-to-date by a community of researchers. Finally, we direct the reader to the collection of exercices [10] by W. Sierpinski.

Marseille, France Olivier Ramaré

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Using this Book for a Course

This chapter is meant for teachers who wish to use (all or part of) this book for a course.

This book may be used from Chaps. 1 to 29 linearly; this gives a long course requiring about 100–120 hours, though it naturally depends on the familiarity of the students with arithmetical notions.

It may also be used for 36-hour courses, which is how these lessons came to be. To build such a course, the *Multiplicativity part* is unavoidable and, in my experience, should always be at least recalled. The *unitary convolution* may be skipped for beginners but is useful to let the more advanced students acquire a new notion.

Once this basic core is over, one may divert to the two light chapters on *Möbius Inversions* and on *Dirichlet Hyperbola Principle*, depending on how much time is available; these chapters may also be kept for term papers.

From there onwards, there are three possibilities: using the *Convolution Method*, applying the *Levin–Faĭnleĭb method*, or invoking the *Mellin Transform*. Some hours should be reserved at the end to delve into one of the last chapters that are grouped in four blocks: the *Selberg Formula* is one, using the *Convolution Method with non-positive multiplicative functions* is the second one and the two heavier ones concerns respectively *Rankin's trick, Brun's sieve and some arithmetical exponential sums*, and the *Montgomery's arithmetical version of the large sieve*. These last lectures will be challenging, all the more so since the students will have less time to acquire the material, but the results therein may be presented at the beginning as the goal of the course. This is how the author has gone about it several times, with his choice of core blocks being

- *Multiplicativity*, followed by the main course of the *Convolution Walk* and the chapter on *Convolution and Non-negative functions*. This gives a light course that can be spiced with some addenda.

- *Multiplicativity*, followed by the main course of the *Levin–Faĭnleĭb Walk* and closing with *Rankin and exponential sums*. This gives a rather strong series on combinatorial methods, showing a path to good results without using the Prime Number Theorem.
- *Multiplicativity*, followed by the main course of the *Mellin Walk* and ending with the *Large sieve/Montgomery's sieve*. This gives a more traditional course.

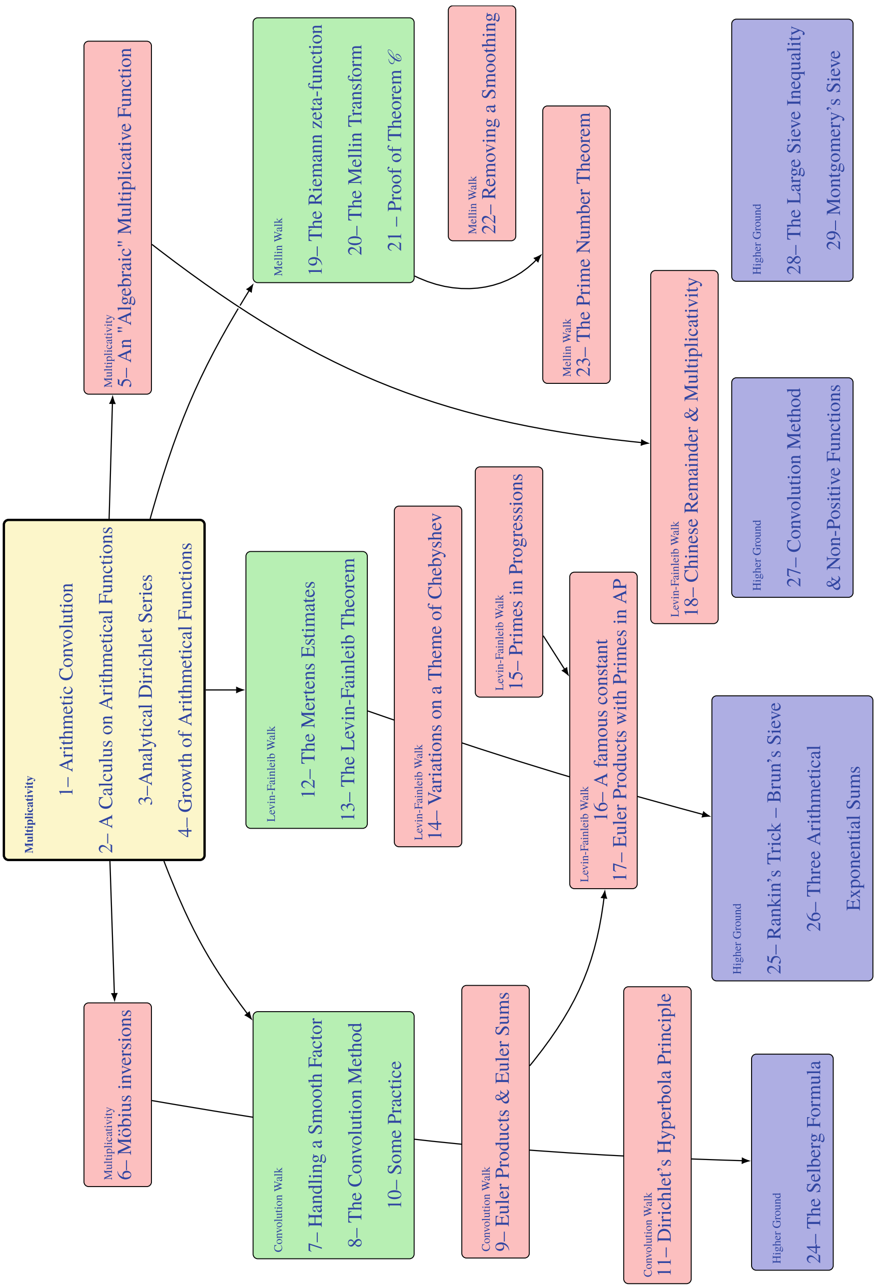
We have tried to represent this architecture in the diagram displayed next page. The core block of *Multiplicativity* is in yellow, the main chapters for each of the three ways are displayed below in green, while the more advanced topics at the bottom of the diagram are being displayed in blue.

Once such a path is chosen, it is possible to dress them with additional material, all displayed in pink. We have ordered them in a way suitable for a full course but there is a large margin of freedom there. It is interesting in the *Levin–Faĭnleĭb Walk* to add information on the distribution of primes in arithmetic progressions, as this extends sizeably the power of this theorem.

Of special interest are the units concerning scientific computations; these are Chaps. 9, 16 and 17. They use the arithmetical background and fit there, but have a different flavour. The arithmetical prerequisite are rather modest, only the following basic definitions are needed: the Euler product representation of the Riemann zeta-function is essential, and later the one concerning Dirichlet series, and finally, the property saying that the Möbius function is the convolution inverse of the constant function $\mathbb{1}$.

The level of difficulty increases from the top to the bottom of the diagram with the chapters not directly depending on their predecessors in the diagram.

To maintain a proper level of independence, some material and often the definitions are retold. We have also tried to be as extensive as we could with cross-references, so that a reader skipping some chapters may still be able to follow.


Multiplicativity
1– Arithmetic Convolution
2– A Calculus on Arithmetical Functions
3–Analytical Dirichlet Series
4– Growth of Arithmetical Functions
Multiplicativity
5– An "Algebraic" Multiplicative Function
Multiplicativity
6– Möbius inversions
Mellin Walk
19– The Riemann zeta-function
20– The Mellin Transform
21 – Proof of Theorem $\mathscr{C}$
Mellin Walk
22– Removing a Smoothing
Mellin Walk
23– The Prime Number Theorem
Levin-Faineib Walk
12– The Mertens Estimates
13– The Levin-Faĭnleib Theorem
Levin-Faĭnleib Walk
14– Variations on a Theme of Chebyshev
Levin-Faĭnleib Walk
15– Primes in Progressions
Levin-Faĭnleib Walk
16– A famous constant
17– Euler Products with Primes in AP
Levin-Faĭnleib Walk
18– Chinese Remainder & Multiplicativity
Convolution Walk
7– Handling a Smooth Factor
8– The Convolution Method
10– Some Practice
Convolution Walk
9– Euler Products & Euler Sums
Convolution Walk
11– Dirichlet's Hyperbola Principle
Higher Ground
24– The Selberg Formula
Higher Ground
25– Rankin's Trick – Brun's Sieve
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