Undergraduate Texts in Mathematics

Béla Bajnok

An Invitation to Abstract Mathematics

Second Edition

Springer

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Preface to Instructors

What Kind of Book Is This?

It has been more than three decades since the first so-called transition book appeared on the mathematics shelves of college bookstores, and there are currently several dozen such books available. The aim of these books is to bridge the gap between the traditional lower-level courses, primarily calculus, and the upper-level courses that require deeper understanding and maturity, such as modern algebra and real analysis. Thus, the main focus of transition books is on the foundations of abstract mathematics, giving a thorough treatment of elementary logic and set theory and introducing students to the art and craft of proof writing.

While this book certainly hopes to provide students with a firm foundation for the upper-level courses of an undergraduate mathematics program, it is not geared solely toward students who intend to major in mathematics. It is the disappointing reality at many institutions that some of the most able students are not considering mathematics as a possible major; in fact, coming out of a standard calculus sequence, most students are not familiar with the true nature of this beautiful subject. Therefore, an important mission of the book is to provide students with an understanding and an appreciation of (abstract) mathematics, with the hope that they choose to study these topics further.

Recognizing that not all our students will have the opportunity to take additional courses in mathematics, this textbook attempts to give a broad view of the field. Even students majoring in mathematics used to complain that they were not given an opportunity to take a course on "mathematics" without an artificial division of subjects. In this textbook, we make an attempt to remedy these concerns by providing a unified approach to a diverse collection of topics, by revisiting concepts and questions repeatedly from differing viewpoints, and by pointing out connections, similarities, and differences among subjects whenever possible. If, during or after reading this book, students choose to take further courses in mathematics, then we have achieved our most important goal.

In order to provide students with a broad exposure to mathematics, we have included an unusually diverse array of topics. Beyond a thorough study of concepts that are expected to be found in similar books, we briefly discuss important milestones in the history of mathematics and feature some of the most interesting recent accomplishments in the field. This book aims to show students that mathematics is a vibrant and dynamic human enterprise by including historical perspectives and notes on the giants of mathematics and their achievements; by mentioning more recent results and updates on a variety of questions of current activity in the mathematical community; and by discussing many famous and less well-known questions that have not yet been resolved and that remain open for the mathematicians of the future.

We also intended to go beyond the typical elementary text by providing a more thorough and deeper treatment whenever feasible. While we find it important not to assume any prerequisites for the book, we attempt to travel further on some of the most enchanting paths than is customary at the beginning level. Although we realize that perhaps not all students are willing to join us on these excursions, we believe that there are a great many students for whom the rewards are worth the effort.

Another important objective—and here is where the author's Hungarian roots get truly revealed—is to center much of the learning on problem-solving. George Pólya's famous book *How to Solve It*[1] introduced students around the world to mathematical problem solving, and, as its editorial review says, "show[ed] anyone in any field how to think straight." Paul Halmos—another mathematician of Hungarian origin[2]—is often quoted[3] about the importance of problems:

> The major part of every meaningful life is the solution of problems; a considerable part of the professional life of technicians, engineers, scientists, etc., is the solution of mathematical problems. It is the duty of all teachers, and of teachers of mathematics in particular, to expose their students to problems much more than to facts.[4]

Our text takes these recommendations to heart by offering a set of carefully chosen, instructive, and challenging problems in each chapter.

It is the author's hope that this book will convince students that mathematics is a wonderful and important achievement of humankind and will generate enough enthusiasm to convince them to take more courses in mathematics. In the process, students should learn how to think, write, and talk abstractly and precisely—skills that will prove immeasurably useful in their future.

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[1] Originally published in hardcover by Princeton University Press in 1945. Available in paperback from Princeton University Press (2004).

[2] Halmos, author of numerous prize-winning books and articles on mathematics and its teaching, is also known as the inventor of the $\square$ symbol, used to mark the end of proofs, and the word "iff," a now-standard abbreviation for the phrase "if, and only if."

[3] For example, by a report of the Mathematical Association of America Committee on the Teaching of Undergraduate Mathematics (Washington, D.C., 1983) and by the *Notices* of the American Mathematical Society (October 2007, page 1141).

[4] "The Heart of Mathematics," *American Mathematical Monthly* 87 (1980), 519–524.

How Can One Teach from This Book?

Can abstract mathematical reasoning be taught? In my view, it certainly can be. However, an honest answer would probably qualify this by saying that not all students will be able (or willing) to acquire this skill to the maximal degree. I often tell people that, when teaching this course, I feel like a ski instructor: I can show them how the pros do it and be there for them when they need my advice, praise, or criticism, but how well they will learn it ultimately depends on their abilities, dedication, and enthusiasm. Some students will be able to handle the steepest slopes and the most dangerous curves, while others will mostly remain on friendlier hills. A few might become Olympic champions, but most will not; however, everyone who gives it an honest effort will at least learn how to move forward without falling. And, perhaps most importantly, I hope that, even though some occasionally find the training frightening and difficult, they will all enjoy the process.

This second edition of the book contains twenty-two chapters and seven appendices arranged in four parts. Each chapter consists of a lecture followed by about a dozen problems. I am a strong believer in the "spiral" method: topics are often discussed repeatedly throughout the book, each time with more depth, additional insights, or different viewpoints. The chapters are written in an increasingly advanced fashion; the last ten chapters (and especially the last three or four) are particularly challenging in both content and language. The lectures and the problems build on one another; the concepts of the lectures are often introduced by problems in previous chapters or are extended and discussed again in problems in subsequent chapters. (The LaTeX command "ref" appears more than one thousand times in the source file.) Therefore, if any part of a lecture or any problem is skipped, this should be done with caution.

The book is designed both for a one-semester course and a two-semester sequence; the latter choice will obviously allow for a more leisurely pace with opportunities for deeper discussions and additional student interactions. The first three parts—What's Mathematics?; The Foundations of Mathematics; and How to Prove It—should probably be covered in any transition course. The last part—Advanced Math for Beginners—aims to give students a fuller view of mathematics. The ten chapters in this part have an approximate dependency chart as follows:

$13 \rightarrow 14 \rightarrow 15$
$16 \rightarrow 17 \rightarrow 18$
$19 \rightarrow 20$
$21 \rightarrow 22$

These four branches are largely independent of one another and can be covered at will.

The heart and soul of this book is in its 329 problems (most with multiple parts); the lectures are intended to be as brief as possible and yet provide enough information for students to attack the problems. I put considerable effort into keeping the number of problems relatively small. Each problem was carefully chosen to clarify a concept, to demonstrate a technique, or to enthuse. There are very few routine problems; most problems will require relatively extensive arguments, creative approaches, or both. Particularly in later chapters, the problems aim for students to develop substantial insight. To make even the most challenging problems accessible to all students, hints are provided liberally.

Many of the problems are followed by remarks aimed at connecting the problems to areas of current research with the hope that some of these notes will invite students to carry out further investigations. The book also contains several appendices with additional material and questions for possible further research. Some of these questions are not difficult, but others require a substantial amount of ingenuity—there are even known open conjectures among them; any progress on these questions would indeed be considered significant and certainly publishable. I feel strongly that every undergraduate student should engage in a research experience. Whether they will go on to graduate school, enroll in professional studies, or take jobs in education, government, or business, students will benefit from the opportunities for perfecting a variety of skills that a research experience provides.

In order to reach a larger audience with this second edition, I made several changes to the first edition. Much of the material was reorganized and streamlined to better match the outline of typical transition courses. In particular, the chapter on functions was moved earlier (and thus received a less abstract treatment); the chapters on mathematical structures moved further back and were augmented by a new separate chapter on groups; and some of the material on the culture, history, and current directions of mathematics were placed into (much enhanced) appendices. This reorganization of course meant that some of the material in other parts of the book had to be rewritten. The second edition includes some of the new developments of the past few years, such as the Ternary Goldbach Conjecture (now theorem) and the improved lower bound on the chromatic number of the plane. Furthermore, to make this admittedly challenging book more approachable, new problems were included and additional guidance (in the form of hints for ways to get started or warnings of potential dead-ends) was added.

Allow me to add a few notes on my personal experiences with this book. I taught courses using this text dozens of times, but find it challenging each time. The approach I find best suited for this course is one that maximizes active learning and class interaction (among students and between students and myself). Students are asked to carefully read the lecture before class and to generate solutions to the assigned problems. Our organized and regular out-of-class "Exploratorium" sessions—where students work alone or with other students in the class under the supervision of teaching associates—seem particularly beneficial in helping students prepare for class. I spend nearly every class by asking students to present the results of their work to the class. As I tell them, it is not necessary that they have completely correct solutions, but I expect them to have worked on all of the problems before class to the best of their ability. I try to be generous with encouragement, praise, and constructive criticism, but I am not satisfied until a thorough and complete solution is presented for each problem. It is not unusual for a problem to be discussed several times before it gets my final PFB ("Perfect for Béla") approval.

Without a doubt, teaching this course has been one of the most satisfying experiences I have had in this profession. Watching my students develop and succeed, perhaps more so than in any other course, is always a superbly rewarding adventure.

Preface to Students

Who Is This Book Written For?

This book is intended for a broad audience. Any student who wishes to learn and perfect his or her ability to think and reason at an advanced level will benefit from taking a course based on this textbook. The skills of understanding and communicating abstract ideas will prove useful in every professional career: law, medicine, engineering, business, education, politics, science, economics, and others. The ability to express oneself and to argue clearly, precisely, and convincingly helps in everyday interactions as well. Just as others can see if we look healthy physically, they can also assess our intellectual fitness when they listen to our explanations or read our writings. Abstract mathematics, perhaps more than any other field, facilitates the learning of these essential skills.

An important goal of this book, therefore, is to help students become more comfortable with abstraction. Paradoxically, the more one understands an abstract topic or idea, the less abstract it will seem! Thus, the author's hope is that his *Invitation to Abstract Mathematics* is accepted, but that, by the time students finish the book, they agree that there is no need for the word "abstract" in the title—indeed, this book is (just) about mathematics.

The prerequisites to the text are minimal; in particular, no specific knowledge beyond high school mathematics is assumed. Instead, students taking this course should be willing to explore unusual and often difficult topics and be ready to face challenges. Facing and overcoming these challenges will be students' ultimate reward at the end.

How Can One Learn from This Book?

Welcome to abstract mathematics! If you are like 99 percent of the students who have taken a course based on this book, you will find that the course is challenging you in ways that you have not been challenged before. Unlike those in your previous mathematics courses, the problems in this book will not ask you to find answers using well-described methods. Instead, you will face problems that you have not seen before, and you will often find yourself puzzled by them for hours, sometimes days. In fact, if you do not need to struggle with the concepts and problems in this book, then this is the wrong course for you since you are not being challenged enough to sharpen your mind. (You should not worry too much about not being challenged though!)

My recommendations to you are as follows. If your instructor gives a lecture, make sure you understand what is being said by asking questions—your classmates will be grateful too. At home, read the relevant material *slowly*. Again, if anything is not completely clear, ask. Next, attack the assigned problems. Do not say that I did not warn you: these problems are hard! In almost all cases, you will need several attempts before you find a solution. It can happen that you spend days without any progress on a given problem, or that the solution you discover at midnight will prove wrong when you try to write it up the next morning. If this happens, *don't panic!* And, *don't give up!*

If you feel you do not know where to start, make sure that you understand what the problem is asking. Look at special cases. Draw illustrative diagrams. Try to turn the question into a simpler question and solve that first. If all these fail, give yourself a break (by moving to another problem) and come back to the beast at a later time.

If you are not sure whether your solution is valid, explain it to others. If they are not convinced or cannot follow your argument, it often means that you have a gap in your proof. Making a jump from one statement to the next without being able to furnish the details means that your work is incomplete.

Even if you succeed in solving a problem, I recommend that you consult with other students in your class. It is always beneficial to discuss your work with others; you might find it interesting to listen to the thoughts of a variety of people and to compare different approaches to the same questions. You might learn more from these brainstorming sessions than by working alone—and, for all but the most antisocial people, it is a lot more fun too!

I am absolutely convinced that your work will pay off, and I hope that you will find it enjoyable as well!

Acknowledgments

I am fortunate to have grown up in a country with a long tradition of superb mathematics education. The legendary culture of mathematics in Hungary[1] was influenced by the many giants who took an active role in education at all levels; for example, Paul Erdős, the most prolific mathematician in history, was a frequent visitor to elementary schools to talk about "adult" mathematics. I remain greatly influenced by the excellent education I received in Hungary, and I am especially grateful to Professors Róbert Freud, Edit Gyarmati, Miklós Laczkovich, and Lajos Pósa who are most responsible for my love of mathematics.

This book is a result of many years of teaching a transition course at Gettysburg College. I would like to thank all the students who used earlier versions of the book during the past—their feedback was invaluable. I am particularly thankful to my colleagues Darren Glass, Benjamin Kennedy, and Keir Lockridge for having used the first edition of this book; the second edition incorporates their many insightful observations and recommendations. I also greatly benefited from the advice on both content and style provided by other friends and colleagues, including Matthias Beck, Steve Berg, Joseph Bonin, Jill Dietz, Róbert Freud, Klaus Peters, Máté Wierdl, and Paul A Zeitz. My students Peter Francis, Emma Gruner, and Hoang Anh Just and freelance editor Martha Masterson Francis offered to carefully read the draft of the second edition—I am very thankful to them for the astute comments and suggestions. I also wish to express my gratitude to my editor Loretta Bartolini for her attentive handling of the manuscript, to the UTM series editors for helpful suggestions, and to everyone else at Springer who assisted with the production of this book.

I would be interested in hearing your opinions, whether you are an instructor, a student, or a casual reader of this book. Also, please let me know of any mistakes or typos. I can be contacted at bbajnok@gettysburg.edu. Thanks!

[1] See, for example, "A Visit to Hungarian Mathematics," by Reuben Hersch and Vera John-Steiner, in *The Mathematical Intelligencer*, Volume 15, (2), (1993) 13–26.

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