Coimbra Mathematical Texts 1

Maria Manuel Clementino ·
Alberto Facchini · Marino Gran *Editors*

New Perspectives in Algebra, Topology and Categories

Summer School, Louvain-la-Neuve, Belgium, September 12–15, 2018 and September 11–14, 2019

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Coimbra Mathematical Texts

Volume 1

Series Editors

Ana Paula Santana, Department of Mathematics, University of Coimbra, Coimbra, Portugal

Júlio S. Neves, Department of Mathematics, University of Coimbra, Coimbra, Portugal

Marcelo Viana, IMPA, Rio de Janeiro, Brazil

Maria Paula Serra de Oliveira, Department of Mathematics, University of Coimbra, Coimbra, Portugal

Rui Loja Fernandes, Department of Mathematics, University of Illinois, Urbana, IL, USA

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Editors
Maria Manuel Clementino
Departamento de Matemática
Universidade de Coimbra
Coimbra, Portugal

Alberto Facchini
Dipartimento di Matematica
Università di Padova
Padova, Italy

Marino Gran
Institut de Recherche en Mathématique
et Physique
Université catholique de Louvain
Louvain-la-Neuve, Belgium

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Editor's Preface

This book is addressed to a new generation of students and researchers in algebra, topology and category theory who are willing to learn the fundamental notions and principles of some of the active research fields in these areas.

The content of each of the seven chapters of this book fully covers the corresponding specialized course that was taught at the international "Summer Schools in Algebra and Topology". The schools were hosted by the Institut de Recherche en Mathématique et Physique (IRMP) of the Université catholique de Louvain (UCLouvain) in September 2018 and 2019.

These events were organized in the framework of a project Fonds d'Appui à l'Internationalisation of the Université catholique de Louvain together with the universities of Coimbra, Padova and Poitiers, aiming at strengthening the research collaborations in mathematics among institutions of the Coimbra Group. The speakers of the summer schools gave some introductory courses on various topics of interest related to their own research fields, addressed to an audience including both master's and PhD students and more experienced mathematicians who were interested in these subjects. The participants in the summer schools came from many different countries in Europe, North America and Asia.

These summer schools greatly facilitated scientific discussions leading to the development of new connections among the various concepts and theories presented during the lectures.

The main subjects in algebra and topology the reader will be introduced to in this volume are:

- the theory of comodules and contramodules;
- the theory of topological semi-abelian algebras;
- the theories of commutative monoids and of non-commutative rings;
- regular categories and the calculus of relations;
- the categorical approach to commutator theory;
- locale theory and the point-free perspective on topology;
- the theory of non-associative algebras.

We are very grateful to the many colleagues who have contributed in making this project possible and successful. In particular, we would like to express our gratitude to

- Ms. Dana Samson, *Prorectrice à l'international de l'UCLouvain*;
- Ms. Ana Fernandez-Gacio, Coordinator of the international relations of the *Science and Technology Sector* of the UCLouvain;
- Ms. Anne Querinjean, director of the *Musée universitaire de Louvain*;
- Ms. Carine Baras, Secrétaire de recherche;
- Ms. Martine Furnemont, Secrétaire of the IRMP;
- Mr. Michel Devillers, Vice-rector of the Science and Technology Sector;
- Mr. Tim van der Linden, Professor at the UCLouvain.
- The editors of the "Coimbra Mathematical Texts", Ana Paula Santana, Júlio S. Neves, Marcelo Viana, Maria Paula Serra de Oliveira and Rui Loja Fernandes.

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- UCLouvain (via the *Fonds d'Appui à l'Internationalisation*)
- The Institut de Recherche en Mathématique et Physique;
- The Centre for Mathematics of the University of Coimbra;
- the *Coimbra Group;*
- the *Ecole doctorale en Mathématique du FNRS;*
- the *Erasmus+* programme of the European Union.

Sincerely,

Maria Manuel Clementino
Alberto Facchini
Marino Gran

Preface to the Book Series Coimbra Mathematical Texts

This new Springer series "Coimbra Mathematical Texts" is a direct successor of a collection entitled "Textos de Matemática", published by the University of Coimbra, which started in 1993.

The aim of that collection was the publication of advanced mathematics texts resulting from events held at the University of Coimbra—Mathematics Department, and included conference proceedings as well as monographs related to graduate-level short courses. The first volume in the collection was "Classical Invariants", by J. A. Green (1993), and the last one, no. 48, was "13th Young Researchers Workshop on Geometry, Mechanics and Control: three mini-courses" (2019). The complete list of volumes, covering a wide range of mathematical topics, may be found at

https://www.uc.pt/fctuc/dmat/seccoes/publicacoes/textosDeMatematica

All volumes went through a rigorous refereeing process.

The series now gains a new life with the launching of "Coimbra Mathematical Texts", a partnership between Springer Nature and the University of Coimbra. The new collection maintains the same spirit but aims at publishing advanced texts from more diverse origins. We therefore invite the wider mathematical community to submit quality contributions to the series, both monographs and proceedings, in all branches of mathematics.

Inquiries regarding the submission process should be sent to cmt@mat.uc.pt.

Ana Paula Santana
Júlio S. Neves
Marcelo Viana
Maria Paula Serra de Oliveira
Rui Loja Fernandes
The CMT Series Editors

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