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Dan Haran
Moshe Jarden

The Absolute Galois Group of a Semi-Local Field

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*In Erinnerung an Wulf-Dieter Geyer
(1939–2019)*

Introduction

The main result of [HJP12], which is Theorem A below, describes the absolute Galois group of distinguished semi-local algebraic extensions of \mathbb{Q} (among others) as free products of \hat{F}_ω and local Galois groups. The proof of Theorem A depends on two results of Florian Pop from [Pop96] and on the main result of [Pop95].

The aim of this monograph is to work out proofs of the above mentioned result of [HJP12] along with the supporting results of Pop. In addition we follow Melnikov's construction in [Mel90] of free products of profinite groups. Finally, we generalize the theory of free products of profinite groups and their subgroups developed in [Har87], and present results appearing in [HJP05] needed in the proofs.

Absolute Galois groups

Our result is an instance of a positive answer to the generalized inverse problem of Galois theory. Originally, this problem asked whether every finite group occurs as a Galois group of a Galois extension of \mathbb{Q} . For many groups this is the case [MaM99], but the general case is still wide open.

One way to realize a finite group over \mathbb{Q} is to do it in pieces. That is, one has to properly solve finite embedding problems over \mathbb{Q} . Again, there are many examples of such problems which are properly solvable [MaM99]. But we do not have a characterization of all finite embedding problems over \mathbb{Q} that are properly solvable. In particular, the structure of the absolute Galois group $\text{Gal}(\mathbb{Q})$ of \mathbb{Q} is unknown.

Still, there are several families of fields with known absolute Galois groups. The most renowned example of a field with this property is $\mathbb{C}(t)$, with t transcendental over \mathbb{C} , or, more generally, finite extensions of $\mathbb{C}(t)$. The Riemann Existence Theorem [Voe96, p. 37, Thm. 2.13] implies that $\text{Gal}(\mathbb{C}(t))$ is the free profinite group on 2^{\aleph_0} generators ([Rib70, p. 70, Thm. 8.1]). An analogous result holds for an arbitrary algebraically closed field of characteristic 0. Various “patching methods” give similar results in the case where K is algebraically closed of positive characteristic (see [Hrb95], [Pop96], or [Jar11]).

By definition, the absolute Galois group of a field K is trivial if K is algebraically closed or, more generally, separably closed.

More subtle is the case where K is the field \mathbb{R} of real numbers or, more generally, real closed. In this case, $\text{Gal}(K)$ is isomorphic to the group with two elements [Lan93, p. 452, Thm. 2.2].

Much more difficult is the case where K is a finite extension of \mathbb{Q}_p or of $\mathbb{F}_p((t))$ for some prime number p . In both cases K is complete with respect to a discrete valuation and $\text{Gal}(K) = T \ltimes W$ is the semi-direct product of its “maximal tame quotient” T and its “wild part” W (Lemma 8.2.2). By Iwasawa, the tame group T is generated by two elements σ, τ satisfying the relation $\sigma\tau\sigma^{-1} = \tau^q$. The wild group W is a free pro- p group of rank \aleph_0 .

For the exact structure of $\text{Gal}(K)$ by generators, relations, and “conditions” we refer the reader to [NSW20, p. 418, Thm. 7.5.13] in the case where $\text{char}(K) = p$ due to Helmut Koch [Koc67] and to [NSW20, p. 419, Thm. 7.5.14] for $\text{char}(K) = 0$ and $p \neq 2$ due to Uwe Jannsen and Kay Wingberg [JaW82]. The case where $\text{char}(K) = 0$ and $p = 2$ was treated by Volker Diekert under the condition that $K(\sqrt{-1})/K$ is unramified. See [NSW20, p. 431] or [Die84].

The field $K_{\text{tot},S}$

From our point of view, more important than the fields \mathbb{Q}_p and $\mathbb{F}_q((t))$ are their algebraic parts. We consider a *classically local prime* \mathfrak{p} of a field K . Thus, the “completion” $\hat{K}_{\mathfrak{p}}$ of K with respect to \mathfrak{p} is either a finite extension of \mathbb{Q}_p or of $\mathbb{F}_q((t))$, where p is a prime number and q is a power of a prime number, or $\hat{K}_{\mathfrak{p}} = \mathbb{R}$. Then, the algebraic part $K_{\mathfrak{p}} = K_{\text{sep}} \cap \hat{K}_{\mathfrak{p}}$ of $\hat{K}_{\mathfrak{p}}$ is the Henselian (respectively, real) closure of K with respect to \mathfrak{p} . This closure is uniquely defined up to K -isomorphism. By a lemma of Krasner (in the Henselian case), $\text{Gal}(K_{\mathfrak{p}})$ is isomorphic to $\text{Gal}(\hat{K}_{\mathfrak{p}})$, so whatever information we have on $\text{Gal}(\hat{K}_{\mathfrak{p}})$ applies also to $\text{Gal}(K_{\mathfrak{p}})$.

This allows us to consider a finite set S of classically local primes of K and set $K_{\text{tot},S} = \bigcap_{\mathfrak{p} \in S} \bigcap_{\rho \in \text{Gal}(K)} K_{\mathfrak{p}}^{\rho}$. By [Pop96, Thm. 3],

$$\text{Gal}(K_{\text{tot},S}) \cong \prod_{\mathfrak{p} \in S} \left[\prod_{\rho \in R_{\mathfrak{p}}} \text{Gal}(K_{\mathfrak{p}})^{\rho} \right].$$

Here, $\prod_{\rho \in R_{\mathfrak{p}}} \text{Gal}(K_{\mathfrak{p}})^{\rho}$ stands for the free product of the profinite groups $\text{Gal}(K_{\mathfrak{p}})^{\rho}$ (Definition 4.1.1), while $\prod_{\mathfrak{p} \in S}$ denotes the free product of finitely many profinite groups following the operator.

We refer to the fields $K_{\text{tot},S}$ as fields of “semi-local type”.

The fields $K_{\text{sep}}(\sigma)$ and $K_{\text{sep}}[\sigma]$

Next we consider fields of another type, “with no arithmetic”. Basic Galois theory shows that the absolute Galois group of a finite field K is isomorphic to $\hat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n\mathbb{Z}$ [FrJ08, p. 15, Sec. 1.5]. It is not difficult to show that the latter property extends

to non-principal ultra products [FrJ08, p. 141, Sec. 7.7] of finite fields. If F is a field of this type and $\text{char}(F) = 0$, then $\text{Gal}(F \cap \tilde{\mathbb{Q}})$ is *procyclic*. Thus, there exists a $\sigma \in \text{Gal}(\mathbb{Q})$ such that $F \cap \tilde{\mathbb{Q}}$ is the fixed field $\tilde{\mathbb{Q}}(\sigma)$ of σ in $\tilde{\mathbb{Q}}$. Conversely, for each $\sigma \in \text{Gal}(\mathbb{Q})$ there exists a non-principal ultraproduct F of finite fields such that $\tilde{\mathbb{Q}}(\sigma) = F \cap \tilde{\mathbb{Q}}$ [Ax67, Thm. 5].

Note that for an arbitrary $\sigma \in \text{Gal}(\mathbb{Q})$ it may happen that $\text{Gal}(\tilde{\mathbb{Q}}(\sigma))$ is not isomorphic to $\hat{\mathbb{Z}}$. For example, this is the case if $\sigma = 1$ or σ is an involution. However, $\text{Gal}(\tilde{\mathbb{Q}}(\sigma)) \cong \hat{\mathbb{Z}}$ for *almost all* $\sigma \in \text{Gal}(\mathbb{Q})$ in the sense of the Haar measure of $\text{Gal}(\mathbb{Q})$ [Ax67, Prop. 3].

The proof of the latter result uses the theory of cyclotomic extensions of \mathbb{Q} . An alternative proof of this theorem applies Hilbert's irreducibility theorem for \mathbb{Q} , hence it holds for every Hilbertian field K . Moreover, the following result holds for every positive integer e and for almost all $\sigma := (\sigma_1, \dots, \sigma_e) \in \text{Gal}(K)^e$: the group $\text{Gal}(K_{\text{sep}}(\sigma))$ is isomorphic to the free profinite group \hat{F}_e on e generators [FrJ08, p. 379, Thm. 18.5.6]. If K is also countable, then for almost all $\sigma \in \text{Gal}(K)^e$ the field $K_{\text{sep}}(\sigma)$ is, in addition, *PAC* [FrJ08, p. 380, Thm. 18.6.1]. This means that every geometrically integral variety over $K_{\text{sep}}(\sigma)$ has a $K_{\text{sep}}(\sigma)$ -rational point.

The latter property implies that the Henselian closures (and the real closures) of almost all fields $K_{\text{sep}}(\sigma)$ are separably closed (a result of Frey–Prestel [FrJ08, p. 205, Cor. 11.5.5]). In this sense, these fields “lack arithmetic”.

Digging further down, we denote the maximal Galois extension of K in $K_{\text{sep}}(\sigma)$ by $K_{\text{sep}}[\sigma]$. Under the latter assumptions on K and e , [FrJ08, p. 669, Thm. 27.4.8] asserts that for almost all $\sigma \in \text{Gal}(K)^e$ the field $K_{\text{sep}}[\sigma]$ is *PAC*, the group $\text{Gal}(K_{\text{sep}}[\sigma])$ is isomorphic to the free profinite group \hat{F}_ω on countably many generators, and $K_{\text{sep}}[\sigma]$ is Hilbertian.

The fields $K_{\text{tot},S}[\sigma]$

As above, we consider a countable Hilbertian field K , a finite set S of classically local primes of K , and a positive integer e . Given $\sigma := (\sigma_1, \dots, \sigma_e) \in \text{Gal}(K)^e$, we consider the fields $K_{\text{tot},S}(\sigma) = K_{\text{tot},S} \cap K_{\text{sep}}(\sigma)$ and $K_{\text{tot},S}[\sigma] = K_{\text{tot},S} \cap K_{\text{sep}}[\sigma]$ of “mixed type”.

Our goal is to reproduce the description of $\text{Gal}(K_{\text{tot},S}[\sigma])$ as it appears in [HJP12, Thm. 3.11] along with all supporting results from [Pop96], [Pop95], [Mel90], and [Har87].

The main result

The main result of this monograph strengthens the main result of [HJP12].

Theorem A (Theorem 9.1.6) *Let K be a countable Hilbertian field, S a finite set of classically local primes of K , and e a positive integer. Then, for almost all $\sigma \in \text{Gal}(K)^e$ the field $K_{\text{tot},S}[\sigma]$ is Hilbertian, PSC, and ample. Moreover, for*

each $\mathfrak{p} \in S$ there exists a closed subset $R_{\mathfrak{p}}$ of $\text{Gal}(K)$ such that $\text{Gal}(K_{\text{tot},S}[\sigma]) \cong \hat{F}_{\omega} * \prod_{\mathfrak{p} \in S} \prod_{\rho \in R_{\mathfrak{p}}} \text{Gal}(K_{\mathfrak{p}})^{\rho}$.

Here an extension M of K in $K_{\text{tot},S}$ is said to be *PSC* if every geometrically integral curve Γ over M with a simple $K_{\mathfrak{p}}^{\rho}$ -point for each $\mathfrak{p} \in S$ and every $\rho \in \text{Gal}(K)$ has infinitely many M -rational points. Also, one says that M is *ample* if every geometrically integral curve over M with a simple M -rational point has infinitely many M -rational points [Jar11, p. 68, Def. 5.3.2].

The Hilbertianity of $K_{\text{tot},S}[\sigma]$ for almost all $\sigma \in \text{Gal}(K)^e$ follows from [BSF13, Thm. 1.1]. By [GeJ02], $K_{\text{tot},S}[\sigma]$ is *PSC* for almost all $\sigma \in \text{Gal}(K)^e$. This implies that M is *ample* [Pop96, Prop. 3.1].

Remark B The free factor $C := \prod_{\mathfrak{p} \in S} \prod_{\rho \in R_{\mathfrak{p}}} \text{Gal}(K_{\mathfrak{p}})^{\rho}$ appearing in Theorem A depends (up to isomorphism) only on K and S but not on the choice of the fields $K_{\mathfrak{p}}$ nor on σ . In particular, this factor is isomorphic to $\text{Gal}(K_{\text{tot},S})$ (Remark 9.2.4).

We call C a *Cantor free product over S* , because each of the spaces $R_{\mathfrak{p}}$ is homeomorphic to the Cantor middle-third set (Section 1.5).

Using the group-theoretic Lemma 4.7.5, Theorem A yields the following corollary.

Corollary C (Remark 9.2.4 and Lemma 9.2.1) *Let K be a countable Hilbertian field, S a finite set of classically local primes of K , and e a non-negative integer. Then, for almost all $\sigma \in \text{Gal}(K)^e$ and for each $\mathfrak{p} \in S$ there exists a closed subset $R_{\mathfrak{p}}$ of $\text{Gal}(K)$ such that*

$$\text{Gal}(K_{\text{tot},S}(\sigma)) \cong \hat{F}_e * \prod_{\mathfrak{p} \in S} \prod_{\rho \in R_{\mathfrak{p}}} \text{Gal}(K_{\mathfrak{p}})^{\rho}.$$

Remark D If S is an empty set, then $C = \mathbf{1}$. Thus, in this case, Corollary C and Theorem A say that for almost all $\sigma \in \text{Gal}(K)^e$ we have $\text{Gal}(K_{\text{sep}}(\sigma)) \cong \hat{F}_e$ and $\text{Gal}(K_{\text{sep}}[\sigma]) \cong \hat{F}_{\omega}$, as mentioned in the Subsection “The fields $K_{\text{sep}}(\sigma)$ and $K_{\text{sep}}[\sigma]$ ”.

A result of Pop

The proof of Theorem A depends on [Pop96, Thm. 2.8]:

Proposition E (The fundamental result, Proposition 8.4.3) *Let S be a finite set of classically local primes of a countable Hilbertian field K . Consider an infinite extension M of K in $K_{\text{tot},S}$ which is ample and Hilbertian. Suppose that $\text{Gal}(M)$ is $\mathcal{G}_{K,S}$ -projective. Then, $\mathcal{G}_{K,S} = \mathcal{G}_{K,S,\max}$ and $\mathcal{G}_{K,S}$ has an étale profinite system \mathcal{R} of representatives for its $\text{Gal}(M)$ -orbits such that $\text{Gal}(M) \cong \hat{F}_{\omega} * \prod_{\Gamma \in \mathcal{R}} \Gamma$.*

Here, $\mathcal{G}_{K,S}$ is the set of all groups $\text{Gal}(K_{\mathfrak{p}})^{\rho}$ with $\mathfrak{p} \in S$ and $\rho \in \text{Gal}(K)$ and the symbol $\mathcal{G}_{K,S,\max}$ stands for the set of all maximal elements of $\mathcal{G}_{K,S}$.

We say that $\text{Gal}(M)$ is $\mathcal{G}_{K,S}$ -projective if every finite $\mathcal{G}_{K,S}$ -embedding problem for $\text{Gal}(M)$ is solvable:

Let G be a profinite group and let \mathcal{G} be a subset of the set of all closed subgroups of G . A *finite \mathcal{G} -embedding problem* for G is a triple $(\varphi: G \rightarrow A, \alpha: B \rightarrow A, \mathcal{B})$, where $\alpha: B \rightarrow A$ is an epimorphism of finite groups, $\varphi: G \rightarrow A$ is a homomorphism of profinite groups, and \mathcal{B} is a set of subgroups of B closed under B -conjugation and taking subgroups, such that for each $\Gamma \in \mathcal{G}$ there exists a homomorphism $\gamma_\Gamma: \Gamma \rightarrow B$ with $\alpha \circ \gamma_\Gamma = \varphi|_\Gamma$ and $\gamma_\Gamma(\Gamma) \in \mathcal{B}$. We say that a homomorphism $\gamma: G \rightarrow B$ is a *solution* if $\alpha \circ \gamma = \varphi$. The solution is *strong* if $\gamma(\Gamma) \in \mathcal{B}$ for each $\Gamma \in \mathcal{G}$. Finally, the embedding problem is *proper* if φ is surjective. In this case, a solution γ to the embedding problem is *proper* if γ is surjective.

We write $\text{Subgr}(G)$ for the set of all closed subgroups of G and equip $\text{Subgr}(G)$ with the *étale topology*. A base for this topology is the family of all open subgroups H of G . A subset \mathcal{R} of $\text{Subgr}(G)$ is said to be *étale profinite* if \mathcal{R} is a profinite space under the induced étale topology of $\text{Subgr}(G)$.

Another result of Pop

The proof of Proposition E depends on the following consequence of a variant of [Pop95, Thm. 3]:

Proposition F (Proposition 6.4.8) *Let G be a profinite group and let \mathcal{G} be a subset of $\text{Subgr}(G)$ of \mathcal{P} -type. Suppose that every finite \mathcal{G} -embedding problem for G has a proper solution.*

Then, every finite \mathcal{G} -embedding problem for G has a proper strong solution.

We do not repeat the definition of \mathcal{P} -type here and only mention that by Definition 6.4.5, $\mathcal{G}_{K,S}$ is of \mathcal{P} -type for every field K and every finite set S of classically local primes of K .

Supporting results

In order to prove Proposition E we also need, in addition to Proposition F, the following result.

Lemma G (Lemma 8.4.2) *Let M be an ample Hilbertian field and let \mathcal{G} be a strictly closed $\text{Gal}(M)$ -invariant subset of $\text{Subgr}(M)$ of \mathcal{P} -type. Suppose that $\text{Gal}(M)$ is \mathcal{G} -projective. Then, every finite proper \mathcal{G} -embedding problem for $\text{Gal}(M)$ has a proper strong solution.*

The proof of Lemma G uses the main Galois-theoretic property of ample fields: Every finite split embedding problem over $M(t)$ with M as in Lemma G and t transcendental over M is properly solvable (see [Pop96, Main Theorem B] or [Jar11, p. 89, Thm. 5.10.2]).

The group-theoretic assumption in Lemma G is satisfied if we assume a stronger field-theoretic assumption on M :

Lemma H (Lemma 8.3.5) *Let M be an infinite field and X a family of separable algebraic extensions of M . Suppose that $\mathcal{G} := \{\text{Gal}(M')\}_{M' \in X}$ is étale compact and M is PXC (Definition 8.3.2). Then, $\text{Gal}(M)$ is \mathcal{G} -projective.*

Generalized Iwasawa isomorphism theorem

In addition to results F and G, the proof of Proposition E uses the following generalization of Iwasawa isomorphism theorem [Pop96, Thm. 4.5]:

Proposition I (Proposition 7.2.2) *Let G and G' be profinite groups. Let \mathcal{G} (resp. \mathcal{G}') be a subset of $\text{Subgr}(G)$ (resp. $\text{Subgr}(G')$) that satisfies the following conditions:*

- (a) $\text{rank}(G) \leq \aleph_0$ (resp. $\text{rank}(G') \leq \aleph_0$).
- (b) \mathcal{G} (resp. \mathcal{G}') is an étale compact set of representatives of the distinct conjugacy classes in $(\mathcal{G}^G)_{\max}$.
- (c) G (resp. G') is properly strongly \mathcal{G} -projective (resp. \mathcal{G}' -projective).
- (d) There is a homeomorphism $\mu: \bigcup_{\Gamma \in \mathcal{G}} \Gamma \rightarrow \bigcup_{\Gamma' \in \mathcal{G}'} \Gamma'$ that satisfies the following condition: for every $\Gamma \in \mathcal{G}$ there is a $\Gamma' \in \mathcal{G}'$ such that $\mu|_{\Gamma}: \Gamma \rightarrow \Gamma'$ is an isomorphism of groups.

Then, there is an isomorphism $\theta: G \rightarrow G'$ such that $\theta(\mathcal{G}^G) = (\mathcal{G}')^{G'}$.

Condition (b) in Proposition I is achieved by the following result.

Lemma J (Lemma 7.1.3) *Let G be a profinite group of rank $\leq \aleph_0$ and let \mathcal{G} be a G -invariant étale compact subset of $\text{Subgr}(G)$ such that $\mathcal{G} = \mathcal{G}_{\max}$ and G is strongly \mathcal{G} -projective. Then, \mathcal{G} has an étale compact subset of representatives for its G -orbits.*

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