UNITEXT 140

Rocco Chirivì · Ilaria Del Corso
Roberto Dvornicich

# Selected Exercises in Algebra

## Volume 2

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# La Matematica per il 3+2

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Roberto Dvornicich

# Selected Exercises in Algebra

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Rocco Chirivì
Department of Mathematics and Physics
University of Salento
Lecce, Italy

Ilaria Del Corso
Department of Mathematics
University of Pisa
Pisa, Italy

Roberto Dvornicich
Department of Mathematics
University of Pisa
Pisa, Italy

*Translated by*
Alessandra Caraceni
Scuola Normale Superiore
Pisa, Italy

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*"The nice thing about mathematics is doing mathematics."*

—Pierre Deligne

*To Andrea, who knows what mathematics is*

Rocco

*To Francesca, with a wish that she will be able to find and nurture her passions*

Ilaria

*To young people who already love or might come to love mathematics*

Roberto

# Preface

This second volume of selected exercises is meant as a companion to the first. Like its predecessor, it contains exam questions set in recent years in the context of the bachelor's degree in mathematics at the University of Pisa. The problems are accompanied by full solutions, a brief overview of the underlying theory, and a series of preliminary exercises.

Our motivation in writing this book and the objectives we hope to achieve with it remain the ones described in the preface to the first volume. Very briefly, these books stem from our conviction that, in order to study and deeply understand algebra (and, indeed, mathematics in general), attending lectures and memorising the statements and proofs of key results are not enough. What one needs to do is *apply* the theory one studies to concrete examples; in practice, use it to solve problems.

This being said, problems come in different varieties. Some require the application of rather simple procedures, or their solution can be derived immediately from definitions and theorem; others do not only require sufficient knowledge of the theory, but also some novel insight by means of which new information is unlocked.

The reader will soon realise that problems of the first type are conspicuously absent from this book; they will be forced to abandon the hope for easy solutions and instead engage with the material at a much deeper level. But *this is how it's done*: mathematics is not a novel to be passively read, but a story to be actively reinvented.

This is why we heartily recommend that the reader arm themselves with patience and do not look up the solution to an exercise before they have spent a considerable amount of time trying to come up with it themselves.

There are a couple more ideas that we would like to stress. The first is that maths enthusiasts at all levels are taken in by the *beauty* of mathematics; the second is that being enthusiastic about anything involves *curiosity* for the truth. We have tried to embody within many of the exercises in this book our personal sense of beauty and the spirit of curiosity that has always inspired us. We hope that the reader might also benefit from this aspect of our work.

As is the case for the first volume, the book is structured according to the way that algebra is taught during the first years of a mathematics degree at the University of

Pisa. When the distinction between bachelor's and master's degrees was introduced, what used to be a single algebra course was split into two parts, which are at present labelled "Arithmetic" and "Algebra 1"; these two parts perfectly correspond to the two volumes of this work.

The "Arithmetic" portion of the material, to which the first volume is devoted, is mostly concerned with basic tools such as induction, some elements of combinatorics, integers and congruences, and includes an introduction to the basic properties of fundamental algebraic structures: Abelian groups, rings, polynomials and their roots, field extensions and finite fields.

The "Algebra 1" portion covered by this second volume delves deeper into group theory, discusses commutative rings with a particular focus on unique factorisation, and deals with field extensions and the fundamentals of Galois theory.

Each part is accompanied by notes on the theory. Although exhaustive, they do not attempt to make up for an algebra textbook, as evidenced by the fact that they contain the statements of relevant results but provide no proofs. (For further information the reader may instead turn, for example, to Herstein's "Topics in Algebra", Wiley & Sons, or Artin's "Algebra", Pearson.)

This book also includes a series of preliminary exercises. These should be tackled first, since they yield results that will often be needed in subsequent problems. We also stress that none of the solutions provided require any theoretical tools other than those recalled in the introductory chapters and the preliminary exercises. Invoking more advanced theorems would simplify certain solutions and occasionally render an exercise trivial, but doing so would be contrary to the very spirit of this work.

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*Updates* We warmly invite readers to provide us with feedback and point out mistakes (which, in a book containing the detailed solutions to hundreds of exercises, are almost inevitably to be found) by contacting us at the addresses: `rocco.chirivi@unisalento.it` or `ilaria.delcorso@unipi.it` or `roberto.dvornicich@unipi.it`.

Updates and corrections will appear on the web page http://www.dmf.unisalento.it/~chirivi/libroEserciziAlgebra.html.

Lecce, Italy — Rocco Chirivì
Pisa, Italy — Ilaria Del Corso
Pisa, Italy — Roberto Dvornicich
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# Contents


**1 Theory** ..... 1
1.1 Groups ..... 1
1.1.1 Basic Notions ..... 1
1.1.2 Homomorphism Theorems ..... 3
1.1.3 Free Groups ..... 4
1.1.4 Presentations of Groups ..... 6
1.1.5 The Dihedral Group ..... 7
1.1.6 Group Automorphisms ..... 8
1.1.7 Commutators and Abelianisation ..... 9
1.1.8 Group Actions ..... 10
1.1.9 The Action by Conjugation ..... 12
1.1.10 The Action by Multiplication ..... 13
1.1.11 $p$-Groups ..... 14
1.1.12 Permutations ..... 15
1.1.13 Abelian Groups ..... 18
1.1.14 Semidirect Product ..... 19
1.1.15 Sylow Theorems ..... 20
1.1.16 Simple Groups ..... 21
1.2 Rings ..... 22
1.2.1 Basic Notions ..... 22
1.2.2 Maximal and Prime Ideals ..... 23
1.2.3 Quotient Rings ..... 24
1.2.4 Ideal Operations ..... 24
1.2.5 Fraction Fields and Localisations ..... 26
1.2.6 Divisibility ..... 27
1.2.7 Euclidean Domains ..... 29
1.2.8 Principal Ideal Domains ..... 31
1.2.9 Unique Factorisation Domains ..... 32
1.2.10 Gaussian Integers ..... 35

1.3 Fields and Galois Theory ..... 37
1.3.1 Basic Notions ..... 37
1.3.2 Homomorphism Extensions ..... 38
1.3.3 Galois Correspondence ..... 41
1.3.4 Cyclotomic Extensions ..... 45
1.3.5 Straightedge and Compass Constructions ..... 47
1.4 Preliminary Exercises ..... 50

**2 Exercises** ..... 89
2.1 Groups ..... 89
2.2 Rings ..... 106
2.3 Fields and Galois Theory ..... 116

**3 Solutions** ..... 125
3.1 Groups ..... 125
3.2 Rings ..... 205
3.3 Fields and Galois theory ..... 243

**Index** ..... 285