

Universitext

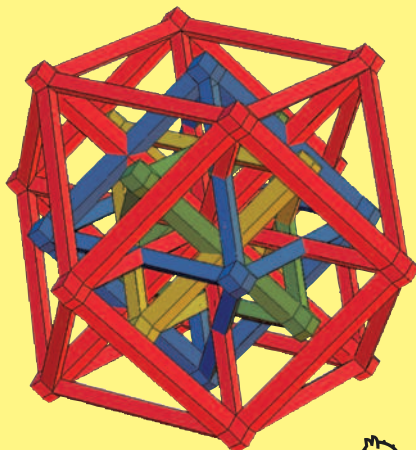
UTX


Yvette Kosmann-Schwarzbach  
Translated by Stephanie Frank Singer

# Groups and Symmetries

From Finite Groups to Lie Groups

*Second Edition*



 Springer

# Universitext

## Series Editors

Carles Casacuberta, Universitat de Barcelona, Barcelona, Spain

John Greenlees, University of Warwick, Coventry, UK

Angus MacIntyre, Queen Mary University of London, London, UK

Claude Sabbah, École Polytechnique, CNRS, Université Paris-Saclay, Palaiseau, France

Endre Süli, University of Oxford, Oxford, UK

*Universitext* is a series of textbooks that presents material from a wide variety of mathematical disciplines at master's level and beyond. The books, often well class-tested by their author, may have an informal, personal even experimental approach to their subject matter. Some of the most successful and established books in the series have evolved through several editions, always following the evolution of teaching curricula, into very polished texts.

Thus as research topics trickle down into graduate-level teaching, first textbooks written for new, cutting-edge courses may make their way into *Universitext*.

More information about this series at <https://link.springer.com/bookseries/223>

Yvette Kosmann-Schwarzbach

# Groups and Symmetries

From Finite Groups to Lie Groups

Second Edition

Translated by Stephanie Frank Singer

 Springer

Yvette Kosmann-Schwarzbach  
Paris, France

*Translated by*  
Stephanie Frank Singer  
Hatfield School of Government  
Portland State University  
Portland, OR, USA

ISSN 0172-5939

ISSN 2191-6675 (electronic)

Universitext

ISBN 978-3-030-94359-2

ISBN 978-3-030-94360-8 (eBook)

<https://doi.org/10.1007/978-3-030-94360-8>

Mathematics Subject Classification: 01-01, 20-01, 20-G05, 20G45, 17B45, 33C55

Original French edition published by Éditions de l'École Polytechnique, 2005.

1<sup>st</sup> edition: © Springer Science+Business Media, LLC 2010

2<sup>nd</sup> edition: © Springer Nature Switzerland AG 2022

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Cover illustration: The figure on the front cover represents weights for the fundamental representation of  $sl(4)$ . Courtesy of Adrian Ocneanu.

This Springer imprint is published by the registered company Springer Nature Switzerland AG  
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland



Sophus Lie (1842–1899), around 1865, at the end of his studies at the University of Christiania (Oslo), approximately seven years before his first work on continuous groups, later known as “Lie groups”.

*(Photo Frederik Klem/Joronn Vogt, with the kind permission of Joronn Vogt and Arild Stubhaug)*

---

# Contents

<b>Introduction</b> . . . . .	xiii
<b>Acknowledgments</b> . . . . .	xix
<b>1 General Facts About Groups</b> . . . . .	1
1 Review of Definitions . . . . .	1
2 Examples of Finite Groups . . . . .	3
2.1 Cyclic Groups . . . . .	3
2.2 Symmetric Groups . . . . .	3
2.3 Dihedral Groups . . . . .	3
2.4 Crystallographic Groups . . . . .	4
3 Examples of Infinite Groups . . . . .	4
4 Group Actions and Conjugacy Classes . . . . .	6
References . . . . .	7
Topics in History . . . . .	7
Exercises . . . . .	10
<b>2 Representations of Finite Groups</b> . . . . .	13
1 Representations . . . . .	13
1.1 General Facts . . . . .	13
1.2 Irreducible Representations . . . . .	15
1.3 Direct Sum of Representations . . . . .	15
1.4 Intertwining Operators and Schur's Lemma . . . . .	16
2 Characters and Orthogonality Relations . . . . .	18
2.1 Matrix Coefficients . . . . .	18
2.2 Characters of Representations and Orthogonality Relations . . . . .	19
2.3 Character Table . . . . .	23
2.4 Application to the Decomposition of Representations . . . . .	23

3	The Regular Representation . . . . .	24
3.1	Definition . . . . .	24
3.2	Character of the Regular Representation . . . . .	25
3.3	Isotypic Decomposition . . . . .	26
3.4	Basis of the Vector Space of Class Functions . . . . .	26
4	Projection Operators . . . . .	28
5	Induced Representations . . . . .	29
5.1	Definition . . . . .	30
5.2	Geometric Interpretation . . . . .	30
	References . . . . .	31
	Topics in History . . . . .	31
	Exercises . . . . .	35
<b>3</b>	<b>Representations of Compact Groups . . . . .</b>	<b>41</b>
1	Compact Groups . . . . .	41
2	Haar Measure . . . . .	42
3	Representations of Topological Groups and Schur's Lemma . . . . .	44
3.1	General Facts . . . . .	44
3.2	Coefficients of a Representation . . . . .	45
3.3	Intertwining Operators . . . . .	45
3.4	Operations on Representations . . . . .	46
3.5	Schur's Lemma . . . . .	47
4	Representations of Compact Groups . . . . .	47
4.1	Complete Reducibility . . . . .	47
4.2	Orthogonality Relations . . . . .	48
5	Summary of Chapter 3 . . . . .	51
	References . . . . .	52
	Topics in History . . . . .	52
	Exercises . . . . .	55
<b>4</b>	<b>Lie Groups and Lie Algebras . . . . .</b>	<b>59</b>
1	Lie Algebras . . . . .	59
1.1	Definition and Examples . . . . .	59
1.2	Morphisms . . . . .	61
1.3	Commutation Relations and Structure Constants . . . . .	61
1.4	Real Forms . . . . .	62
1.5	Representations of Lie Algebras . . . . .	62
2	Review of the Exponential Map . . . . .	64
3	One-Parameter Subgroups of $GL(n, \mathbb{K})$ . . . . .	66
4	Lie Groups . . . . .	68
5	The Lie Algebra of a Lie Group . . . . .	69



6	The Connected Component of the Identity . . . . .	72
7	Morphisms of Lie Groups and of Lie Algebras . . . . .	73
7.1	Differential of a Lie Group Morphism . . . . .	73
7.2	Differential of a Lie Group Representation . . . . .	75
7.3	The Adjoint Representation . . . . .	77
	References . . . . .	79
	Topics in History . . . . .	79
	Exercises . . . . .	83
<b>5</b>	<b>Lie Groups <math>SU(2)</math> and <math>SO(3)</math></b> . . . . .	<b>89</b>
1	The Lie Algebras $\mathfrak{su}(2)$ and $\mathfrak{so}(3)$ . . . . .	89
1.1	Bases of $\mathfrak{su}(2)$ . . . . .	89
1.2	Bases of $\mathfrak{so}(3)$ . . . . .	91
1.3	Bases of $\mathfrak{sl}(2, \mathbb{C})$ . . . . .	92
2	The Covering Morphism of $SU(2)$ onto $SO(3)$ . . . . .	92
2.1	The Lie Group $SO(3)$ . . . . .	93
2.2	The Lie Group $SU(2)$ . . . . .	95
2.3	Projection of $SU(2)$ onto $SO(3)$ . . . . .	96
	References . . . . .	97
	Topics in History . . . . .	97
	Exercises . . . . .	101
<b>6</b>	<b>Representations of <math>SU(2)</math> and <math>SO(3)</math></b> . . . . .	<b>103</b>
1	Irreducible Representations of $\mathfrak{sl}(2, \mathbb{C})$ . . . . .	103
1.1	The Representations $D^j$ . . . . .	103
1.2	The Casimir Operator . . . . .	106
1.3	Hermitian Nature of the Operators $J_3$ and $J^2$ . . . . .	106
2	Representations of $SU(2)$ . . . . .	108
2.1	The Representations $\mathcal{D}^j$ . . . . .	108
2.2	Characters of the Representations $\mathcal{D}^j$ . . . . .	111
3	Representations of $SO(3)$ . . . . .	112
	References . . . . .	113
	Topics in History . . . . .	113
	Exercises . . . . .	116
<b>7</b>	<b>Spherical Harmonics</b> . . . . .	<b>119</b>
1	Review of $L^2(S^2)$ . . . . .	119
2	Harmonic Polynomials . . . . .	120
2.1	Representations of Groups on Function Spaces . . . . .	120
2.2	Spaces of Harmonic Polynomials . . . . .	120
2.3	Representations of $SO(3)$ on Spaces of Harmonic Polynomials . . . . .	121
3	Definition of Spherical Harmonics . . . . .	123

3.1	Representations of $SO(3)$ on Spaces of Spherical Harmonics . . . . .	123
3.2	The Casimir Operator . . . . .	125
3.3	Eigenfunctions of the Casimir Operator . . . . .	125
3.4	Bases of the Spaces of Spherical Harmonics . . . . .	126
3.5	Explicit Formulas . . . . .	129
	References . . . . .	130
	Topics in History . . . . .	130
	Exercises . . . . .	133
<b>8</b>	<b>Representations of <math>SU(3)</math> and Quarks</b> . . . . .	<b>135</b>
1	Representations of $\mathfrak{sl}(3, \mathbb{C})$ and $SU(3)$ . . . . .	135
1.1	Review of $\mathfrak{sl}(n, \mathbb{C})$ . . . . .	135
1.2	The Case of $\mathfrak{sl}(3, \mathbb{C})$ . . . . .	136
1.3	The Bases $(I_3, Y)$ and $(I_3, T_8)$ of $\mathfrak{h}$ . . . . .	138
1.4	Representations of $\mathfrak{sl}(3, \mathbb{C})$ and $SU(3)$ . . . . .	138
2	The Adjoint Representation and Roots . . . . .	138
3	The Fundamental Representation and Its Dual . . . . .	140
3.1	The Fundamental Representation . . . . .	140
3.2	The Dual of the Fundamental Representation . . . . .	141
4	Highest Weight of a Finite-Dimensional Representation . . . . .	142
4.1	Highest Weight . . . . .	142
4.2	Weights as Linear Combinations of the $\lambda_i$ . . . . .	143
4.3	Finite-Dimensional Representations and Weights . . . . .	144
4.4	Another Example: the Representation 6 . . . . .	145
4.5	One More Example: the Representation 10 . . . . .	146
5	Tensor Products of Representations . . . . .	147
6	The Eightfold Way . . . . .	150
6.1	Baryons ( $B = 1$ ) . . . . .	151
6.2	Mesons ( $B = 0$ ) . . . . .	151
6.3	Baryon Resonances . . . . .	152
7	Quarks and Antiquarks . . . . .	153
	References . . . . .	154
	Topics in History . . . . .	154
	Exercises . . . . .	157
<b>9</b>	<b>Spin Groups and Spinors</b> . . . . .	<b>161</b>
1	Clifford Algebras . . . . .	161
1.1	Definition . . . . .	161
1.2	Universal Property . . . . .	162
1.3	Complex and Real Clifford Algebras . . . . .	162
2	The Groups $\text{Pin}(n)$ and $\text{Spin}(n)$ . . . . .	163
2.1	The Group $\text{Pin}(n)$ . . . . .	163

2.2	Adjunction and Conjugation . . . . .	164
2.3	Orthogonal Transformations are Products of Reflections . . . . .	164
2.4	The Group Morphism from $\text{Pin}(n)$ to $\text{O}(n)$ . . . . .	166
2.5	Definition and Properties of the Group $\text{Spin}(n)$ . . . . .	166
2.6	The Groups $\text{Spin}(1)$ , $\text{Spin}(2)$ , and $\text{Spin}(3)$ . . . . .	168
3	Spinor Representations of the Clifford Algebras . . . . .	168
3.1	Representations of Algebras . . . . .	168
3.2	Spinor Representations of the Complex Clifford Algebras . . . . .	169
3.3	The Real Case . . . . .	170
4	Representations of the Spin Groups . . . . .	170
4.1	The Complex Spin Groups . . . . .	170
4.2	The Groups $\text{Spin}(p, q)$ . . . . .	171
4.3	Representations of the Spin Groups and Spinors . . . . .	171
4.4	Spinors in 3 Dimensions . . . . .	172
4.5	Spinors in 4 Dimensions and the Dirac Equation . . . . .	172
4.6	Important Remark . . . . .	173
	References . . . . .	173
	Topics in History . . . . .	173
	Exercises . . . . .	176
	<b>Problems and Solutions</b> . . . . .	179
1	Restriction of a Representation to a Finite Group . . . . .	179
2	The Group $\text{O}(2)$ . . . . .	182
3	Representations of the Dihedral and Quaternion Groups . . . . .	185
4	Representations of $\text{SU}(2)$ and of $\mathfrak{S}_3$ . . . . .	194
5	Pseudo-Unitary and Pseudo-Orthogonal Groups . . . . .	198
6	Irreducible Representations of $\text{SU}(2) \times \text{SU}(2)$ . . . . .	204
7	Projection Operators . . . . .	212
8	Symmetries of Fullerene Molecules . . . . .	219
9	Matrix Coefficients and Spherical Harmonics . . . . .	230
	<b>Endnote</b> . . . . .	237
	<b>Bibliography</b> . . . . .	241
	<b>Index</b> . . . . .	247