Studies in Universal Logic

Hajnal Andréka Zalán Gyenis István Németi Ildikó Sain

Universal Algebraic Logic Dedicated to the Unity of Science







Studies in Universal Logic

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Universal Algebraic Logic

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Preface

Algebraic logic (AL) is a subject of mathematics which connects logical and algebraical ways of thinking. Ideally, it shows how concrete phenomena can be described equivalently both in algebraic terms and in terms of logic.

The idea of solving problems in logic by first translating them to algebra, then using the powerful methodology of algebra for solving them, and then translating the solution back to logic, goes back to Leibnitz and Pascal. Papers on the history of Logic (e.g. Anellis–Houser [40], Maddux [139], Pratt [179]) point out that this method was fruitfully applied in the 19th century not only to propositional logics but also to quantifier logics (De Morgan, Peirce, etc. applied this method to quantifier logics, too). The number of applications grew ever since. For brevity, we will refer to the above method or procedure as "applying Algebraic Logic to Logic".

In items (i) and (ii) below we describe two of the main motivations for applying AL to Logic.

(i) This is the more obvious one: When working with a relatively new kind of problem, it has often proved to be useful to "transform" the problem into a well understood and streamlined area of mathematics, solve the problem there and translate the result back. Examples include the method of Laplace Transform in solving differential equations (a central tool in Electrical Engineering).¹ To transform logical problems to algebra, we define the algebraic counterpart $Alg(\mathcal{L})$ of a logic \mathcal{L} . Then we prove equivalence theorems, which to essential logical properties of \mathcal{L} associate natural and well investigated properties of $Alg(\mathcal{L})$ such that if we want to decide whether \mathcal{L} has a certain property, we will know what to ask from our algebraist colleague about $Alg(\mathcal{L})$. The same devices are suitable for finding out what one has to change in \mathcal{L} if we want to have a variant of \mathcal{L} having a desirable property (which \mathcal{L} lacks). For all this, first we have to define

¹ This general method is discussed in considerable detail in Madarász [133], under the keyword "duality theories". Cf. e.g. Appendix A therein which describes algebraic logic as a special duality theory.

what we understand by a logic \mathcal{L} in general (because otherwise it is impossible to define e.g. the function Alg associating a class $Alg(\mathcal{L})$ of algebras to each logic \mathcal{L}). To apply AL to Logic, one has to travel the bridge both ways since the algebraic solutions have to be translated back to logic. This, of course, enables one to also use the bridge-theorems for solving algebraic problems via tools of logic.

(ii) With the rapidly growing variety of applications of logic (in diverse areas like computer science, linguistics, artificial intelligence, law, the logic of spacetime, relativity theory etc.) there is a growing number of new logics to be investigated. In this situation AL offers a tool for economy and a tool for unification in various ways. One of these is that $Alg(\mathcal{L})$ is always a class of algebras, therefore we can apply the same machinery namely Universal Algebra to study all the new logics. In other words we bring all the various logics to a kind of "normal form" where they can be studied by uniform methods. Moreover, for most choices of \mathcal{L} , $Alg(\mathcal{L})$ tends to appear in the same "area" of Universal Algebra, hence specialized powerful methods lend themselves to studying \mathcal{L} . There is a fairly well understood "map" available for the landscape of Universal Algebra. By using our algebraization process and equivalence theorems we can project this "map" back to the (far less understood) landscape of possible logics.

The title of the book is Universal Algebraic Logic. One reading of this title is (Universal Algebraic) Logic. This refers to a stance of algebraic logic where the algebraic part is universal algebra (since the logic part is universal logic, too). The other reading of the title is Universal (Algebraic Logic). This second reading refers to the fact that the methods in the book are applicable to all kinds of logics, not only to a selected choice of them. Universal Algebraic Logic (UAL) is closely related to Abstract Algebraic Logic (AAL). Both the aims and methods are very similar. While in UAL there is more emphasis on the semantical aspects of a logic and on easy applicability, AAL strives for elegance of the concepts. A novelty in the present book is that methods for treating quantifier logics (e.g., first-order logic FOL) in their original, non-substitutional forms are elaborated. Emphasis on the semantical aspects and treating FOL connect UAL to the subjects of model theoretical logics and the theory of institutions. Section 3.4 contains a more detailed comparison between AAL, UAL, model theoretic logics and the theory of institutions. Perhaps the most important direction in UAL missing is the treatment of many-sorted logics. Starting points in this direction might be [66] and [192]. Many-sorted logics are more and more important in applying logic to other sciences, see e.g., Halvorson [93]. We recommend [93] also for roots of AL in the Vienna Circle, and about the sub-title of our book "Dedicated to the unity of science".

The present book consists of the following chapters.

Chapter 1 introduces the basic notation used in the book, and at the same time, it presents a brief informal reminder/introduction of the set theoretic foundations.

Chapter 2 is a brief self contained introduction to Universal Algebra. Universal algebra is often called abstract algebra or general algebra. We chose the adjective "universal" to indicate that we treat the subject as a general theory of complex structures as opposed to a kind of distillation of earlier branches of algebra.

Chapter 3 describes the method of translating logical problems to algebra and back. We bring the various logics to a kind of "normal form" in section 3.1. Then, in section 3.3, to each logic \mathcal{L} presented in this normal form we define the algebraic counterpart $\operatorname{Alg}_{\models}(\mathcal{L})$ of the syntactical part of \mathcal{L} together with the algebraic counterpart $\operatorname{Alg}_m(\mathcal{L})$ of the semantical-model theoretical ingredients of \mathcal{L} . Section 3.2 contains a large number of different examples for logics brought in the normal form and section 3.4 compares our formalism with those of abstract algebraic logic, model theoretical logics and theory of institutions.

Chapter 4 contains some bridge-theorems. These connect natural logical properties of \mathcal{L} to natural algebraic properties of $\mathsf{Alg}_{\models}(\mathcal{L})$ and $\mathsf{Alg}_m(\mathcal{L})$. Instead of a thorough treatment, we wanted to illustrate the main ideas and techniques. In the first four subsequent sections we connect to the logical properties compactness, completeness, definability and interpolation properties the algebraic properties of being closed under ultraproducts, finite axiomatizability, surjectivity of epimorphisms and amalgamation properties, respectively. Properties of category theoretical flavor of $\mathsf{Alg}_{\models}(\mathcal{L})$ and $\mathsf{Alg}_{m}(\mathcal{L})$ keep showing up, especially when treating first-order like logics. We conclude each of these sections with applications of the theorems both for solving logical problems via algebraic methods and for solving algebraic problems via logical methods, to illustrate that the bridge can be walked in two ways. In the last two sections we touch upon decidability issues and Gödel incompleteness properties. Some of the issues already elaborated in the literature but not presented here are: omitting types property (e.g., [193]), various model theoretic properties like saturated models universal models etc (see, e.g., [198], [199], [169]), interpretations and definitional equivalence between theories (e.g., [97, sec.4.3]).

The eight sections of chapter 5 contain eight short case-studies for applying the methods of universal algebraic logic. Classical first-order logic is one of them.

Chapter 6 contains some more relaxed properties of logics and logics with different flavor than in section 3.2.

Finally, chapter 7 contains the definitions and most basic properties of the classes of algebras occurring in chapter 5. It also contains some proofs that show the usefulness of the universal algebraic methods, in particular the methods applicable to discriminator varieties. In some sense, this chapter can be viewed as a very short introduction to Tarskian algebraic logic.

* * *

The approach reported here is strongly related to works of Blok and Pigozzi cf. e.g. [46], [48], [47], [177], Czelakowski [61], Font–Jansana [69]. Strongly related material by the present authors appeared in [33] and [173]. To keep the present work self-contained, we had to repeat here some of the things we wrote there. Another strongly related paper is Andréka–Németi–Sain–Kurucz [35], using a somewhat different setting. That setting gives a broader perspective, however, the investigations of Hilbert-style inference systems done herein are not yet pushed through in that setting. A common root of all the work mentioned in this paragraph is Henkin–Monk–Tarski [97] §5.6. Applications and other extensions of the present work include [218], [39], [16], [17], [172], [173], [186], [189], [190], [3], [133].

This book grew out of course materials, like [34], [32] (cf. also [31]) used first at the Logic Graduate School, Budapest, at the beginning of the 1990s. Therefore our style often remained that of a lecturer writing to her/his students.

Teaching the book. This book, apart from being a monograph containing state of the art results in algebraic logic, can be used as the basis for a number of different courses intended for both novices and more experienced students of logic, mathematics, or philosophy. Below we list some suggestions.

Universal algebra. (13 weeks, 90 minutes a week)

- 1. Elementary concepts (ch. 1): direct products (sec. 1.5) and closure systems (sec. 1.7).
- 2. Warming up examples (sec. 2.1) and lattices (sec. 2.2.3).
- 3. Subalgebras (sec. 2.2.1) and homomorphisms (sec. 2.2.2).
- 4. Congruences (sec. 2.2.4) with an emphasis on Theorem 2.2.34.
- 5. Direct products (sec. 2.2.5), direct decomposition of finite algebras (Thm. 2.2.46).
- Subdirect decomposition (sec. 2.2.6), Birkhoff's subdirect decomposition theorem (Thm. 2.2.58).
- 7. Ultrafilters, ultraproducts (sec. 2.2.7), Łoś ultraproduct theorem (Thm. 2.2.74).
- 8. Free algebras (sec. 2.5) and the beginning of varieties (sec. 2.4).
- 9. Variety characterization (Thm. 2.4.11), quasi-variety characterization (Thm. 2.4.12).
- 10. Boolean algebras (sec. 2.6).
- 11. Discriminator varieties (sec. 2.7).
- 12. Boolean algebras with operators (sec. 2.8).
- 13. Selected applications from section 7.1.

Algebraic logic (basic track). (13 weeks, 90 minutes a week)

1. Defining the framework for studying logics (sec. 3.1); Definitions 3.1.3, 3.1.5 and 3.1.6. Selected examples from section 3.2 with an emphasis on propositional logic (def. 3.2.3) and finite variable first order logic (def. 3.2.19).

- 2. Basics from universal algebra: examples (sec. 2.1) and subalgebras (sec. 2.2.1).
- 3. Homomorphisms (sec. 2.2.2), congruences (sec. 2.2.4), Theorem 2.2.34.
- 4. Connectives (sec. 3.3.1), compositionality, tautological formula algebra (sec. 3.3.2), Algebraic counterparts of logics (sec. 3.3.3) until Theorem 3.3.5.
- Direct products (sec. 2.2.5), direct decomposition of finite algebras (Thm. 2.2.46), subdirect decomposition (sec. 2.2.6), Birkhoff's subdirect decomposition theorem without proof (Thm. 2.2.58).
- 6. Algebraization applied to propositional logic (def. 3.2.3) and its connection with Boolean (set) algebras (sec. 2.6).
- Substitution properties (sec. 3.3.4) until the characterization of semantical substitution property (def. 3.3.11). Filter property (sec. 3.3.5), Theorem 3.3.23.
- General logics (sec. 3.3.6) until the characterization of substitutional general logics (prop. 3.3.32). Examples and Theorem 3.3.36.
- 9. Compactness properties (def. 4.1.1 and 4.1.4). Ultraproducts and Loś's theorem without proof (sec. 2.2.7 and Thm. 2.2.74). Algebraic characterization of compactness: Theorem 4.1.5, and Theorem 4.1.6 without proof.
- Hilbert-type inference systems (sec. 4.2.1), algebraization of admissible rules (Thm. 4.2.5). Definition of complete and sound Hilbert-type inference systems (def. 4.2.7).
- 11. Algebraic characterization of finitely complete and strongly sound inference systems (Thm. 4.2.9). Deduction term (def. 4.2.16) and Theorem 4.2.18.
- 12. Interpolation properties with examples (sec. 4.4.1). Amalgamation (def. 4.4.17) and Theorem 4.4.18 (without proof). The superamalgamation property (def. 4.4.23) and the algebraic characterization of interpolation (Thm. 4.4.24).
- 13. Selected applications from chapter 5.

Algebraic logic (advanced track). (13 weeks, 180 minutes a week)

This course presupposes a solid background in universal algebra and firstorder logic. The material consists of all of section 3.1, selected examples from section 3.2, all of 3.3 except for the proofs involving the conditional substitution property; and all of chapter 4. It is a demanding course but we think that students with the required background are able to master the mentioned topics in one semester.

Concerning the figures in the book: Our teaching experience indicates that the majority of students appreciate drawings very much. Drawings support them in their understanding as well as in memorizing the material. This is why we included some drawings. We encourage our readers to prepare more drawings for themselves when reading this book.

To help the instructor to make up a course, we give below a dependency diagram of the sections. This diagram applies to the main bulk of the material.



Fig. 0.1 Diagram of dependency

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