Algorithms and Computation in Mathematics 26

[ACM]

Oleg N. Karpenkov

# Geometry of Continued Fractions

*Second Edition*

Springer

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Volume 26

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# Preface to the Second Edition

The idea of the second edition was originally motivated by improvement of certain notation within the chapters and correcting various typos suggested by the readers. However during this work I decided to add several interesting theorems that were missing in the first edition. For the convenience of the readers who are familiar with the first edition I would like to underline here the main changes that were made. My intention was not to overload the book with new topics but rather to improve the exposition of the existing ones.

- First of all in Section 1.3 we relate partial numerators and partial denominators to the classical notion of continuants. We supplement numerous formulae via expressions in terms of continuants further in the text.
- We have added a criterion of rational angles congruence in Subsection 2.1.8 and of integer triangle convergence (Proposition 6.7).
- In the new Section 2.5 and Section 18.6 we show the classification of integer-regular polygons and polyhedra respectively.
- We have included an explicit expression for LLS sequences of adjacent angles in terms of certain long continued fractions (see Section 5.5).
- Two algorithms to compute LLS sequences are added to Chapter 4 (see Section 4.5).

Finally, the chapter on Gauss Reduction Theory (Chapter 7 of the first edition of the book) was a subject of the major metamorphose. It was substantially revised and split into several new chapters:

- Markov numbers are discussed in a separate Chapter 7 now.
- The section on geometry of continued fractions is substantially extended to new Chapter 8. In particular we have added a new technique of computation of LLS sequence periods for $\mathrm{GL}(2,\mathbb{Z})$ matrices.
- Chapter 9 on continuant representations of $\mathrm{GL}(2,\mathbb{Z})$ matrices is new. It is very much in the spirit of Gauss Reduction Theory.

- The semigroup of reduced matrices is discussed separately in Chapter 10.
- The remaining material (of Chapter 7 of the first edition of the book) is now placed in Chapter 11: here we have added proofs for elliptic and parabolic matrices and revised the main case of the hyperbolic matrices. Additionally we have extended the exposition to the group $\mathrm{GL}(2,\mathbb{Z})$ (originally it was mostly regarding $\mathrm{SL}(2,\mathbb{Z})$).

Further examples and exercises were added to different chapters of the book.

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*Oleg Karpenkov*
February 2022

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# Preface to the First Edition

Continued fractions appear in many different branches of mathematics: the theory of Diophantine approximations, algebraic number theory, coding theory, toric geometry, dynamical systems, ergodic theory, topology, etc. One of the metamathematical explanations of this phenomenon is based on an interesting structure of the set of real numbers endowed with two operations: addition $a+b$ and inversion $1/b$. This structure appeared for the first time in the Euclidean algorithm, which was known several thousand years ago. Similarly to the structures of fields and rings (with operations of addition $a+b$ and multiplication $a*b$), structures with addition and inversion can be found in many branches of mathematics. That is the reason why continued fractions can be encountered far away from number theory. In particular, continued fractions have a geometric interpretation in terms of integer geometry, which we place as a cornerstone for this book.

The main goal of the first part of the book is to explore geometric ideas behind regular continued fractions. On the one hand, we present geometrical interpretation of classical theorems, such as the Gauss—Kuzmin theorem on the distribution of elements of continued fractions, Lagrange's theorem on the periodicity of continued fractions, and the algorithm of Gaussian reduction. On the other hand, we present some recent results related to toric geometry and the first steps of integer trigonometry of lattices. The first part is rather elementary and will be interesting for both students in mathematics and researchers. This part is a result of a series of lecture courses at the Graz University of Technology (Austria). The material is appropriate for master's and doctoral students who already have basic knowledge of linear algebra, algebraic number theory, and measure theory. Several chapters demand certain experience in differential and algebraic geometry. Nevertheless, I believe that it is possible for strong bachelor's students as well to understand this material.

In the second part of the book we study an integer geometric generalization of continued fractions to the multidimensional case. Such a generalization was first considered by F. Klein in 1895. Later, this subject was almost completely abandoned due to the computational complexity of the structure involved in the calculation of the generalized continued fractions. The interest in Klein's generalization was revived by V.I. Arnold approximately one hundred years after its invention, when computers became strong enough to overcome the computational complexity. After a brief introduction to multidimensional integer geometry, we study essentially new questions for the multidimensional cases and questions arising as extensions of the classical ones (such as Lagrange's theorem and Gauss—Kuzmin statistics). This part is an exposition of recent results in this area. We emphasize that the majority of examples and even certain statements of this part are on two-dimensional continued fractions. The situation in higher dimensions is more technical and less studied, and in many cases we formulate the corresponding problems and conjectures. The second part is intended mostly for researchers in the fields of algebraic number theory, Diophantine equations and approximations, and algebraic geometry. Several chapters of this part can be added to a course for master's or doctoral students.

Finally, I should mention many other interesting generalizations of continued fractions, coming from algorithmic, dynamical, and approximation properties of continued fractions. These generalizations are all distinct in higher dimensions. We briefly describe the most famous of them in Chapter 27.

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February 2013

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