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Yuichiro Hoshi
Shinichi Mochizuki

Topics Surrounding the Combinatorial Anabelian Geometry of Hyperbolic Curves II

Tripods and Combinatorial Cuspidalization

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Yuichiro Hoshi
RIMS
Kyoto University
Kyoto, Japan

Shinichi Mochizuki
RIMS
Kyoto University
Kyoto, Japan

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Preface

Let Σ be a subset of the set of prime numbers which is either equal to the entire set of prime numbers or of cardinality one. In the present book, we continue our study of the pro-Σ fundamental groups of hyperbolic curves and their associated configuration spaces over algebraically closed fields in which the primes of Σ are invertible. The starting point of the theory of the present book is a *combinatorial anabelian result* which, unlike results obtained in previous papers, allows one to *eliminate* the hypothesis that *cuspidal inertia subgroups* are *preserved* by the isomorphism in question. This result allows us to [partially] generalize **combinatorial cuspidalization** results obtained in previous papers to the case of outer automorphisms of pro-Σ fundamental groups of configuration spaces that *do not necessarily preserve the cuspidal inertia subgroups* of the various one-dimensional subquotients of such a fundamental group. Such partial combinatorial cuspidalization results allow one in effect to reduce issues concerning the **anabelian geometry** of **configuration spaces** to issues concerning the anabelian geometry of **hyperbolic curves**. These results also allow us, in the case of configuration spaces of sufficiently large dimension, to give **purely group-theoretic** characterizations of the **cuspidal inertia subgroups** of the various one-dimensional subquotients of the pro-Σ fundamental group of a configuration space. We then turn to the study of **tripod synchronization**, i.e., roughly speaking, the phenomenon that an outer automorphism of the pro-Σ fundamental group of a log configuration space associated to a stable log curve typically induces the **same** outer automorphism on the various subquotients of such a fundamental group determined by **tripods** [i.e., copies of the projective line minus three points]. Our study of tripod synchronization allows us to show that outer automorphisms of *pro-*Σ fundamental groups of configuration spaces exhibit somewhat **different behavior** from the behavior that may be observed—as a consequence of the classical **Dehn-Nielsen-Baer theorem**—in the case of *discrete* fundamental groups. Other applications of the theory of tripod synchronization include a result concerning **commuting profinite Dehn multi-twists** that, a priori, arise from distinct *semi-graphs of anabelioids of pro-*Σ *PSC-type* structures [i.e., the profinite analogue of the notion of a *decomposition of a hyperbolic topological surface into hyperbolic subsurfaces*, such as "pants"], as well

as the computation, in terms of a certain **scheme-theoretic fundamental group**, of the *purely combinatorial/group-theoretic commensurator* of the group of **profinite Dehn multi-twists**. Finally, we show that the condition that an outer automorphism of the pro-Σ fundamental group of a stable log curve *lift* to an outer automorphism of the pro-Σ fundamental group of the corresponding n-th log configuration space, where $n \geq 2$ is an integer, is compatible, in a suitable sense, with **localization** on the dual graph of the stable log curve. This localizability property, together with the theory of tripod synchronization, is applied to construct a **purely combinatorial analogue** of the natural outer **surjection** from the étale fundamental group of the moduli stack of hyperbolic curves over $\mathbb{Q}$ to the **absolute Galois group** of $\mathbb{Q}$.

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