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Convex Analysis and Beyond

Volume I: Basic Theory

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Volume I: Basic Theory

With 42 Figures

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To the memory of

JON BORWEIN (1951–2016)

and

DIETHARD PALLASCHKE (1940–2020),

our dear friends

and prominent researchers in convex analysis and beyond

Preface

Convexity has a long history that could date back to Ancient Greece geometers. Probably the first definition of convexity was given by Archimedes of Syracuse in the third century BC: *There are in a plane certain terminated bent lines, which either lie wholly on the same side of the straight lines joining their extremities, or have not part of them on the other side.*

Over the subsequent centuries, convexity theory has been developed in various *geometric frameworks* with outstanding contributions of many great mathematicians. The most active period in the study of convex sets was in the late nineteenth century and the early twentieth century with the quintessential work done by Hermann Minkowski that was summarized in the books [219, 220] published in 1910–1911 after his death. Minkowski laid the foundations of the general theory of convex sets in finite-dimensional spaces. In particular, he established there the fundamental *convex separation theorem*, which since has played a crucial role in convex analysis and its applications.

A systematical study of *convex functions* has been started by Werner Fenchel who discovered, in particular, seminal results on *conjugacy* correspondence and *convex duality* contained in his very influential mimeographed lecture notes [131] from a course given at Princeton University in 1951.

Although some constructions and results on generalized differentiation of convex functions can be found in Fenchel [131], the fundamental notion of *subdifferential* (a collection of subgradients) for an extended-real-valued convex function should be attributed to Jean-Jacques Moreau [264] and R. Tyrrell Rockafellar [302], who introduced this notion independently in 1963. The revolutionary idea of a *set-valued* generalized derivative satisfying rich *calculus* rules has given rise to *convex analysis*, a new area of mathematics where analytic and geometric ideas are so nicely interrelated and jointly produce beautiful results for sets, set-valued mappings, and functions.

A milestone in the consolidation of the new discipline, at least in finite-dimensional spaces, was Rockafellar's monograph "Convex Analysis" [306] published in 1970, which coined the name of this area of mathematics. Over the subsequent years, numerous strong results have been discovered in this area and many excellent books have been published on various aspects of convex analysis and its applications in finite and infinite dimensions. Among them, we mention the books by Bauschke and Combettes [34], Bertsekas et al. [37], Borwein and Lewis [48], Boyd and Vandenberghe [62], Castaing and Valadier [71], Ekeland and Temam [122], Hiriart-Urruty and Lemaréchal [164] and its abridge version [165], Ioffe and Tikhomirov [174], Nesterov [279], Pallaschke and Rolewicz [285], Phelps [290], Pshenichnyi [294], and Zălinescu [361].

It has been well recognized that convex analysis provides the mathematical foundations for numerous applications in which convex optimization is the first to name. The presence of convexity makes it possible not only to investigate qualitative properties of optimal solutions and derive efficient optimality conditions, but also develop and justify numerical algorithms to solve convex optimization problems with smooth and nonsmooth data. Convex analysis and optimization have an increasing impact on many areas of mathematics and applications including control systems, estimation and signal processing, communications and networks, electronic circuit design, data analysis and modeling, statistics, economics and finance, etc. In recent times, convex analysis has become more and more important for applications to some new fields of mathematical sciences and practical modeling such as computational statistics, machine learning, sparse optimization, location sciences, etc.

Despite an extensive literature on diverse aspects of convex analysis and applications, our book has a lot to offer to researchers, students, and practitioners in these fascinating areas. We split the book into *two volumes*, and now present to the reader's attention the *first volume*, which is mainly devoted to theoretical aspects of convex analysis and related fields where convexity plays a crucial role. The *second volume* [240] addresses various applications of convex analysis including those areas, which were listed above.

The first volume is devoted to developing a *unified theory* of convex sets, set-valued mappings, and functions in vector and topological vector spaces with its specifications to Banach and finite-dimensional settings. These developments and expositions are based on the powerful *geometric approach* of *variational analysis*, which resides on *set extremality* with its characterizations and significant modifications in the presence of convexity. This approach allows us to consolidate the device of fundamental facts of generalized differential calculus and obtain novel results for convex sets, set-valued mappings, and functions in finite-dimensional and infinite-dimensional settings.

Some aspects of the geometric approach to convex analysis in finite-dimensional spaces were given in our previous short book [237] in which the reader was provided with an easy path to access generalized differentiation of convex objects in finite dimensions and its applications to theoretical and algorithmic topics of convex optimization and facility location. Now we largely extend in various directions the previous developments in both finite-dimensional and infinite-dimensional spaces while covering a much broader spectrum of topics and results related to convexity and its applications.

Besides major topics of convex analysis, we present in this book several important developments, which have been either motivated by the extensions of their convexity prototypes, or being largely based on convexity methods and results. The first group includes *variational principles* of Ekeland's type for lower semicontinuous functions that are strongly related to Bishop-Phelps's density theorems and their proofs for nonsolid convex sets. The second group concerns *convexified nonsmooth analysis* of nonconvex functions admitting *convex* generalized directional derivatives for which calculus rules and other properties are based on methods and results of convex analysis. All of this allows us to title our book as "Convex Analysis and Beyond."

The book consists of *seven chapters* and is organized as follows. *Chapter* 1 addresses the *mathematical foundations* of convex analysis. For the reader's convenience, we make the book *self-contained* and present here basic concepts and results on topological spaces and topological vector spaces scattered in the literature and give their short while rather detailed proofs. The selected concepts and results are used further to study algebraic, geometric, and topological properties of convex sets and functions. Also, this chapter includes the fundamental theorems of functional analysis largely employed in the book, accompanied by their simplified albeit complete proofs.

Chapter 2 is devoted to *basic theory of convexity* for sets and functions in linear spaces and topological vector spaces with some specifications in finite dimensions. We pay special attention to *convex sets* and derive for them various versions of *convex separation theorems*, which play a pivoting role in further developments. For extended-real-valued functions, the convexity is defined geometrically with giving analytical representations, describing operations over functions that keep convexity, and studying major topological properties of convex functions. The last section of this chapter contains a more recent material on *generalized relative interiors* of convex sets in infinite-dimensional spaces, including new results in this direction.

In *Chapter* 3 we start the exposition and development of a unified theory of *generalized differentiation* for convex sets, set-valued mappings, and extended-real-valued functions based on the aforementioned variational geometric approach. We mainly concentrate here on the general setting of *topological vector spaces* with presenting also important finite-dimensional specifications. The essence of our approach is the notion of *set extremality* and the corresponding *convex extremal principle* for systems of sets, which goes along with the general extremal principle of *variational analysis* while significantly reflecting the specific of convex sets and being closely related to convex separation. The main calculus result obtained is the *normal cone intersection rule* derived under various qualification conditions. Based on this result, we establish comprehensive rules for coderivatives of set-valued mappings and subgradients of extended-real-valued functions in topological vector space and finite-dimensional settings with special elaborations for classes of maximum and distance functions. Finally, in this chapter, we present recent and new results on *polyhedral* calculus rules in topological vector spaces that essentially extend their prototypes in finite dimensions.

Chapter 4 enters the consideration of *Fenchel conjugates* of extended-real-valued functions, which are among the strongest tools of convex analysis and are closely related to *convex duality*. Based on the previous study of *set extremality* and *convex separation*, we first develop here a comprehensive *conjugate calculus* under appropriate qualification conditions in the three major settings: in general topological vector spaces with using nonempty *interiors* of sets and the *continuity* of functions, under *polyhedrality* assumptions in topological vector spaces with using *quasi-relative interiors* of sets, and in *finite dimensions* with the usage of *relative interiors*. Furthermore, *enhanced rules* of conjugate and generalized differential calculus are developed in this chapter under relaxed qualification conditions by employing *variational techniques* in *Banach spaces*. A special attention is paid to subdifferentiation of the *pointwise suprema* of convex functions over infinite sets with the usage of relationships between subgradients and *directional derivatives*, and to computing subgradients and conjugates of *marginal/optimal value functions*, which are highly important for numerous applications. Finally, *Chapter* 4 presents major developments on *Fenchel duality* including quite recent and new results on this topic in various space frameworks and under diverse qualification conditions.

Chapter 5 contains complete proofs of major *variational principles* of variational analysis and their convex counterparts. We highlight, in particular, novel *approximate* and *exact* versions of the *extremal principle* for closed convex sets in general Banach spaces. These results give *necessary and sufficient conditions* for *set extremality* that are different in several aspects from the corresponding versions of the extremal principle in nonconvex settings given in the book of Mordukhovich [228]. As a consequence of the extremal principle, we establish new approximate and exact characterizations of *convex separation* for closed convex subsets of Banach spaces without any (generalized) relative interiority assumptions. Among the topics presented in this chapter, where the developed variational principles and arguments play a large role, we mention *calculus of* ε*-subgradients* with and without qualification conditions, *mean value theorems* for continuous and lower semicontinuous convex functions, *maximal monotonicity* of subgradient mappings, and subdifferential characterizations of *Gâteaux and Fréchet differentiability* together with their *generic* versions. This chapter is concluded by considering matrix-dependent *spectral* and *singular functions* with giving a simple proof of the seminal *von Neumann trace inequality* and the subsequent subdifferential study of these functional classes, which are highly important in applications.

In *Chapter* 6 we address *miscellaneous topics* of convex analysis that combine classical results with more recent themes that play a crucial role in a variety of theoretical and algorithmic applications to practical models considered in the second volume of our book. Classical topics include *Carathéodory, Helly* and *Radon theorems*, *Farkas lemma*, *duality relationships* between tangent and normal cones and their calculations for *polyhedral sets*, *horizon cones* and *horizon functions* at infinity, etc. More recent developments concern *Nesterov's smoothing techniques* and related topics on *strong convexity* and *strong monotonicity* in finite and infinite dimensions, the study of *perspective functions* at infinity as well as of *signed distance* and *minimal time functions* together with the new class of *signed minimal time functions*. Although the main emphasis in the study of these classes of functions is on their convex analysis, some important results do not require any convexity assumptions.

The final *Chapter* 7 addresses *nondifferentiable* while *nonconvex* functions that is different from the previous chapters. However, the study of such functions is mainly based on the machinery of *convex analysis*. This really goes *beyond* convex analysis by using certain *convexification* procedures. The major attention is paid in this chapter to the *parallel* investigation of two convex-valued *directionally generated* subdifferentials of locally Lipschitzian functions on normed spaces that are associated with Clarke's *generalized directional derivative* and the Dini *contingent derivative/subderivative*. We present comprehensive calculus rules and other results for these and related constructions including quite recent and new developments. Some properties of *regular* and *limiting subgradients* are also reviewed here with their usage in deriving precise subdifferential formulas for the signed distance functions associated with convex sets. Finally, we include major results for a very important class of *DC* (*difference of convex*) functions with applications to *nonconvex duality*.

Each chapter contains the *exercise section*, where we formulate numerous exercises with different levels of difficulties and provide hints to some of them. Many *figures* and *examples* are given throughout the whole text. Furthermore, the last section of each chapter presents extensive *commentaries*, which play a highly significant role in the book. Besides detailed historical information and reviewing the genesis of major ideas and motivations, we provide in the commentaries rather elaborated discussions on some relates topics, which are not included in the basic text; e.g., relationships with similar results of nonconvex variational analysis, subdifferentiation of integral functionals, directionally generated subgradient mappings for nonconvex extended-real-valued functions, etc. All of this makes the book to be more complete and leads the reader to additional advanced studies.

Different parts of this book aim at their primary groups of readers. The entire book should be of interest for experts in convex and variational analysis, optimization, and their numerous applications as well as for mathematicians and applied scientists in other areas who wish to learn more on this subject. Based on our own experience in teaching some parts of this book at Portland State University and Wayne State University, we envision that the book with the exercises therein will be useful for teaching graduate classes in mathematical sciences that are also accessible to advanced students in economics, engineering, and other applied areas. Large parts of the book concerning convex analysis in finite-dimensional and normed spaces present well accessible material for upper undergraduate students.

Over the years of our work on the book, we have enjoyed fruitful discussions with many prominent experts in convex and variational analysis, optimization, and their applications whose publications are included in the reference list. Our special thanks go to Terry Rockafellar, to the late Jon Borwein and Diethard Pallaschke, and also to Yurii Nesterov, Nguyen Dong Yen, and Constantin Zălinescu. We are very grateful to all our collaborators of the papers and projects that are used in the book. Our students Anuj Bajaj, Liam Jemison, Scott Lindstrorm, Will Maxwell, Dao Nguyen, Trang Nguyen, Nguyen Xuan Quy, and Gary Sandine helped us in the book preparation and proofreading. Finally, both authors thank the National Science Foundation, while the first author also thanks the Air Force Office of Scientific Research for their continuing support.

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