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Optimal Design of Multi-Phase Materials

With a Cost Functional That Depends Nonlinearly on The Gradient

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To my wife Patrocinio and my daughter Ana.

Preface

Since the 60s, it is well known that the microscopical mixture (or more exactly mesoscopic mixture) of two or more materials allows us to obtain other materials whose behavior is very different from the original. This does not only depend on the proportion of each material used in the mixture but also on their geometrical dispositions in the mixture. The branch of mathematics dealing with this problem is the homogenization theory that began with the pioneering works of E. De Giorgi, Z. Hashin, F. Murat, E. Sánchez-Palencia, S. Shtrikman, S. Spagnolo, L. Tartar,... Among his achievements, we mention a general compactness result showing that the limit of the solutions of a sequence of second-order linear elliptic equations with uniformly bounded, and uniformly coercive coefficients converge to the solution of a problem of the same type. It allows us to describe the new materials as the limit of a sequence of mixtures of the original ones. These results have been extended to the elasticity system, evolutive equations, equations with unbounded and/or non-uniformly elliptic coefficients, nonlinear equations,...

One of the main applications of these results is the study of the optimal arrangement of several materials in order to minimize a certain functional. In this sense we recall that some counterexamples due to F. Murat show that the problem has no solution in general and therefore the need of working with a relaxed formulation. The main idea to get this relaxed formulation is to deal with general mixtures obtained through the limit process mentioned above. Moreover, the relaxed formulation is easier to handle from the point of view of the numerical analysis due to its convexity and derivability properties.

Our purpose in this work is to carry out an introduction to the application of the homogenization theory to the optimal design of diffusion materials (electric of thermic) obtained as the mixture of some given components (multi-phase materials). In the corresponding control problem, these materials are represented through the coefficients of a second-order linear elliptic equation. The functional to minimize depends on the proportion of the materials used in the mixture and the solution to the partial differential problem. This has been carried out in some very good bibliographies such as G. Allaire's book: shape optimization by the homogenization Method, Springer, 2002 and the references therein. Here, we are mainly interested in the case of a functional, which is not continuous with respect to the state and flux functions in the weak topology of $H_0^1(\Omega) \times L^2(\Omega)^N$. This is the case of functionals that depend nonlinearly of the gradient of the state function. It causes that the functional changes its form in the relaxed formulation. We show that it still admits an integral representation but it is not explicit in general. However, we get some interesting properties such as a partial convexity and an explicit formula for the value of the relaxed functional in the boundary of its domain. Assuming further conditions, we also get upper and lower bounds and even in very particular cases, a representation in the whole of its domain.

A major difficulty arising from not having an explicit representation of the relaxed functional is the numerical resolution of the problem. We present some strategies, which use an upper or a lower approximation of the functional. We show the convergence of discrete versions of the problem where the relaxed materials are constant in the elements of a mesh, and the corresponding Sobolev space is replaced by a Galerkin approximation. Assuming some smoothness conditions on the relaxed functional (or more exactly in the approximation used in the numerical method), which can be proved to hold in several important examples, we provide a descend algorithm based on the convexity properties of the control set. The smoothness assumptions on the functional also allow us to get a system of optimality conditions and to deduce some qualitative properties of the solutions.

In the last chapter of the book, we make a short introduction to the case of multistate problems, i.e. to the case where there are more than one state equation. From the application point of view, this means that we want to construct a material which works well in several situations. Now, even in the case of functionals that do not depend on the gradient, we find the difficulty that a complete description of the set of materials obtained by the homogenization process is only known in very few cases. The most important example is the mixture of two isotropic materials. In the case of a unique state equation, only a partial knowledge of this set is necessary. If the functional depends nonlinearly on the gradient of the state functions, and there are an infinite number of state equations, then it is not clear that the relaxed functional admits an integral representation. This is due to a lack of compactness. Similar considerations also hold for the case of evolutive problems that we briefly discuss.

A more detailed description of the content of each chapter is carried out in its corresponding abstract.

Each chapter contains a bibliographic section with references to previous works.

Most of the new results in the present book are a continuation of previous works and discussions with other colleagues: J. Couce-Calvo, J. Castro, F. Murat, J. D. Martín-Gómez, M. Luna-Laynez, E. Zuazua. My sincere thanks to all of them.

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Seville, Spain — Juan Casado-Díaz
January 2022

Notations

- Ω denotes a bounded open set of $\mathbb{R}^N$.
- $C_c^k(\Omega)$ with $0 \le k \le \infty$ denotes the space of functions of class C^k in Ω with compact support. $C_0^k(\Omega)$ is the closure of $C_c^k(\Omega)$ in $C^k(\overline{\Omega})$.
- $L^p(\Omega)$, with $1 \le p \le \infty$ denotes the usual Lebesgue spaces.
- $W^{1,p}(\Omega)$ denotes the space of functions in $L^p(\Omega)$ with distributional derivative in $L^p(\Omega)$. The closure of $C_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$ is denoted by $W_0^{1,p}(\Omega)$. In the case $p = 2$ we use the notations $H^1(\Omega) = W^{1,2}(\Omega)$ and $H_0^1(\Omega) = W_0^{1,2}(\Omega)$.
- $\mathscr{M}(\Omega)$ denotes the space of bounded Radon measures in Ω. It can be identified with the dual $C_0^0(\Omega)$. $\mathscr{M}(\overline{\Omega})$ denotes the space of Radon measures in $\overline{\Omega}$. It can be identified with the dual $C^0(\overline{\Omega})$.
- For a set $\omega \subset \mathbb{R}^N$, we denote by χ_ω the characteristic function of ω.
- The Lebesgue measure of a set $\omega \subset \mathbb{R}^N$ is denoted by $|\omega|$.
- The open ball of center x and radius $r > 0$ in $\mathbb{R}^N$ is denoted by $B(x, r)$. For the closed ball we use $\overline{B}(x, r)$.
- The unit sphere in $\mathbb{R}^N$ is denoted by S_{N-1}.
- For a vector $\xi \in \mathbb{R}^N$ we denote by ξ_i its i-th component. Analogously the coefficients of a matrix M are denoted by M_{ij}.
- The scalar product of two vectors $\xi, \eta \in \mathbb{R}^N$ is denoted as $\xi \cdot \eta = \sum_{i=1}^N \xi_i \eta_i$. The scalar product of two matrix $M, L \in \mathbb{R}^{N\times N}$ is denoted by $M : L = \sum_{i,j=1}^N M_{ij} L_{ij}$.
- For two vectors $\xi, \eta \in \mathbb{R}^N$, we write $\xi \le \eta$ if $\xi_i \le \eta_i$, $1 \le i \le N$.
- For two symmetric matrices $A, B \in \mathbb{R}^{N\times N}$, we write $A \le B$ if $B - A$ is nonnegative.
- The tensorial product $\xi \otimes \eta$ of two vectors $\xi, \eta \in \mathbb{R}^N$ is the matrix in $\mathbb{R}^{N\times N}$ of coefficients $(\xi \otimes \eta)_{ij} = \xi_i \eta_j$. The symmetric tensorial product of ξ, η is defined by $\xi \odot \eta = (\xi \otimes \eta + \eta \otimes \xi)/2$.
- The transposed of a matrix M is denoted by M^t.
- The kernel and the range of a matrix M are respectively denoted by $\mathrm{Ker}(M)$ and $\mathrm{Ran}(M)$.
- The spectrum of a matrix $M \in \mathbb{R}^{N\times N}$ is denoted by $\mathrm{Sp}(M)$.
- $\mathscr{O}^+$ is the set of the orthogonal matrices in $\mathbb{R}^N$ with determinant equals to 1.
- $\{e_1, \ldots, e_m\}$ is the canonical basis of $\mathbb{R}^m$.

- T_m is the convex hull of $\{e_1, \dots, e_m\}$, *i.e.*

$$T_m = \{(p_1, \dots, p_m) \in \mathbb{R}^m : 0 \le p_i, p_1 + \cdots + p_m = 1\}.$$

- Y is the unitary cube in $\mathbb{R}^N$, $Y = (0, 1)^N$.
- The index $\sharp$ means periodicity. For example, $L^p_\sharp(Y)$ is the space of functions in $L^p_{loc}(\mathbb{R}^N)$ which are periodic of period Y.
- For a set X and a function $F : X \to (-\infty, \infty]$, we define the domain of F and we denote it by $D(F)$, as the set of $x \in X$ such that $F(x) < \infty$.

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