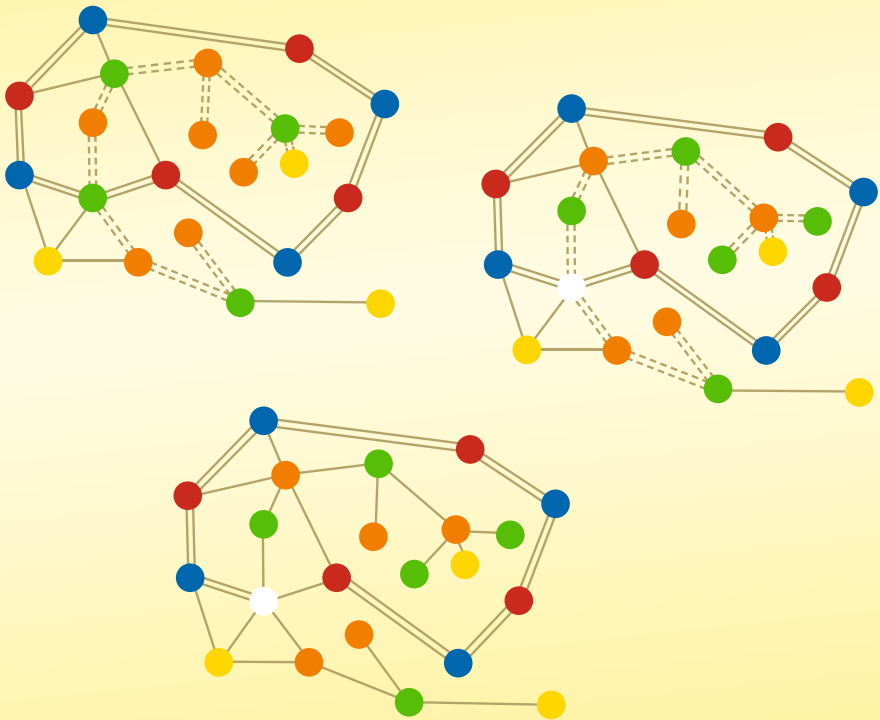


Mordechai Ben-Ari

# Mathematical Surprises



OPEN ACCESS

 Springer

# Mathematical Surprises

Mordechai Ben-Ari

# Mathematical Surprises

 Springer

Mordechai Ben-Ari  
Department of Science Teaching  
Weizmann Institute of Science  
Rehovot, Israel



ISBN 978-3-031-13565-1      ISBN 978-3-031-13566-8 (eBook)  
<https://doi.org/10.1007/978-3-031-13566-8>

Mathematics Subject Classification (2020): 00-01, 51Nxx, 05-01, 00A07

© The Editor(s) (if applicable) and The Author(s) 2022. This book is an open access publication.

**Open Access** This book is licensed under the terms of the Creative Commons Attribution 4.0 International License (<http://creativecommons.org/licenses/by/4.0/>), which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons license and indicate if changes were made.

The images or other third party material in this book are included in the book's Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the book's Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors, and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG  
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

# Foreword

If everyone were exposed to mathematics in its natural state, with all the challenging fun and surprises that that entails, I think we would see a dramatic change both in the attitude of students toward mathematics, and in our conception of what it means to be “good at math.”

Paul Lockhart

I'm really hungry for surprises because each one makes us ever-so-slightly but substantially smarter.

Tadashi Tokieda

Mathematics, when appropriately approached, can provide us with plentiful pleasant surprises. This is confirmed by a Google search of “mathematical surprises,” which, surprisingly, yields almost half a billion items. What is a surprise? The origins of the word trace back to Old French with roots in Latin: “sur” (over) and “prendre” (to take, to grasp, to seize). Literally, to surprise is to overtake. As a noun, surprise is both an unanticipated or bewildering event or circumstance, as well as the emotion caused by it.

Consider, for example, an extract from a lecture by Maxim Bruckheimer<sup>1</sup> on the Feuerbach circle: “Two points lie on one and only one straight line, this is no surprise. However, three points are not necessarily on one straight line and if, during a geometrical exploration, three points ‘fall into’ a straight line, this is a surprise and frequently we need to refer to this fact as a theorem to be proven. Any three points not on a straight line lie on one circle. However, if four points lie on the same circle, this is a surprise that should be formulated as a theorem. . . . Insofar as the number of points on a straight line is larger than 3, so is the theorem the more surprising.

---

<sup>1</sup> Maxim Bruckheimer was a mathematician who was one of the founders of the Open University UK and Dean of its Faculty of Mathematics. He was Head of the Department of Science Teaching at the Weizmann Institute of Science.

Likewise, insofar as the number of points lying on one circle is larger than 4, so is the theorem the more surprising. Thus, the statement that for any triangle there are nine related points on the same circle . . . is very surprising. Moreover, in spite of the magnitude of the surprise, its proof is elegant and easy.”

In this book Mordechai Ben-Ari offers a rich collection of mathematical surprises, most of them less well known than the Feuerbach Circle and with sound reasons for including them. First, in spite of being absent from textbooks, the mathematical gems of this book are accessible with just a high school background (and patience, and paper and pencil, since fun does not come for free). Second, when a mathematical result challenges what we take for granted, we are indeed surprised (Chaps. 1, 13). Similarly, we are surprised by: the cleverness of an argument (Chaps. 2, 3), the justification of the possibility of a geometric construction by algebraic means (Chap. 16), a proof relying on an apparently unrelated topic (Chaps. 4, 5), a strange proof by induction (Chap. 6), new ways of looking at a well-known result (Chap. 7), a seemingly minor theorem becoming the foundation of a whole field of mathematics (Chap. 8), unexpected sources of inspiration (Chap. 9), rich formalizations emerging from purely recreational activities such as origami (Chaps. 10–12). These are all different reasons for the inclusion of the pleasant, beautiful and memorable mathematical surprises in this lovely book.

So far I have addressed how the book relates to the first part of the definition of surprise, the cognitive rational reasons for the unexpected. As to the second aspect, the emotional aspect, this book is a vivid instantiation of what many mathematicians claim regarding the primary reason for doing mathematics: it is fascinating! Moreover, they claim that mathematics stimulates both our intellectual curiosity and our esthetic sensibilities, and that solving a problem or understanding a concept provides a spiritual reward, which entices us to keep working on more problems and concepts.

It has been said that the function of a foreword tell readers why they should read the book. I have tried to accomplish this, but I believe that the fuller answer will come from you, the reader, after reading it and experiencing what the etymology of the word surprise suggests: to be overtaken by it!

*Abraham Arcavi*

# Preface

Godfried Toussaint's article on the "collapsing compass" [50] made a profound impression on me. It would never have occurred to me that the modern compass with a friction joint is not the one used in Euclid's day. In this book I present a selection of mathematical results that are not only interesting, but that surprised me when I first encountered them.

The mathematics required to read the book is secondary-school mathematics, but that does not mean that the material is simple. Some of the proofs are quite long and require that the reader be willing to persevere in studying the material. The reward is understanding of some of the most beautiful results in mathematics. The book is not a textbook, because the wide range of topics covered doesn't fit neatly into a syllabus. It is appropriate for enrichment activities for secondary-school students, for college-level seminars and for mathematics teachers.

The chapters can be read independently. (An exception is that Chap. 10 on the axioms of origami is a prerequisite for Chaps. 11, 12, the other chapters on origami.) Notes relevant to all chapters are given below in list labeled Style.

## What Is a Surprise?

There were three criteria for including a topic in the book:

- The theorem surprised me. Particularly surprising were the theorems on constructibility with a straightedge and compass. The extremely rich mathematics of origami was almost shocking: when a mathematics teacher proposed a project on origami, I initially turned her down because I doubted that there could be any serious mathematics associated with the art form. Other topics were included because, although I knew the results, their proofs were surprising in their ele-

gance and accessibility, in particular, Gauss's purely algebraic proof that a regular heptadecagon can be constructed.

- The material does not appear in secondary-school and college textbooks, and I found these theorem and proofs only in advanced textbooks and in the research literature. There are Wikipedia articles on most of the topics, but you have to know where to look and the articles are often outlines.
- The theorems and proofs are accessible with a good knowledge of secondary-school mathematics.

Each chapter concludes with a paragraph *What Is the Surprise?* which explains my choice of the topic.

## An Overview of the Contents

Chapter 1 presents Euclid's proof that any construction that is possible with a fixed compass is possible with a collapsing compass. Many proofs have been given, but, as Toussaint shows, most are incorrect because they depend on diagrams which do not always correctly depict the geometry. To emphasize that one must not trust diagrams, I present the famous alleged proof that every triangle is isosceles.

Over the centuries mathematicians unsuccessfully sought to trisect an arbitrary angle (divide it into three equal parts) using only a straightedge and compass. Underwood Dudley made a comprehensive study of trisectors who find incorrect constructions; most constructions are approximations that are claimed to be accurate. Chapter 2 starts by presenting two of these constructions and developing the trigonometric formulas showing that they are only approximations. To show that trisection using just a straightedge and compass is of no practical importance, trisections using more complex tools are presented: Archimedes's *neusis* and Hippias's *quadratrix*. The chapter ends with a proof that it is impossible to trisect an arbitrary angle with a straightedge and compass.

Squaring a circle (given a circle construct a square with the same area) cannot be performed using a straightedge and compass, because the value of  $\pi$  cannot be constructed. Chapter 3 presents three elegant constructions of close approximations to  $\pi$ , one by Kočański and two by Ramanujan. The chapter concludes by showing that a quadratrix can be used to square a circle.

The four-color theorem states that it is possible to color any planar map with four colors, such that no countries with a common boundary are colored with the same color. The proof of this theorem is extremely complicated, but the proof of the five-color theorem is elementary and elegant, as shown in Chapter 4. The chapter also presents Percy Heawood's demonstration that Alfred Kempe's "proof" of the four-color theorem is incorrect.



How many guards must be employed by an art museum so that all the walls are under constant observation by at least one guard? The proof in Chapter 5 is quite clever, using graph coloring to solve what at first sight appears to be a purely geometrical problem.

Chapter 6 presents some lesser-known results and their proofs by induction: theorems on Fibonacci numbers and Fermat numbers, McCarthy's 91 function, and the Josephus problem.

Chapter 7 discusses Po-Shen Loh's method of solving quadratic equations. The method is a critical element of Gauss's algebraic proof that a heptadecagon can be constructed (Chapter 16). The chapter includes al-Khwarizmi's geometric construction for finding roots of quadratic equations and a geometric construction used by Cardano in the development of the formula for finding roots of cubic equations.

Ramsey theory is a topic in combinatorics that is an active area of research. It looks for patterns among subsets of large sets. Chapter 8 presents simple examples of Schur triples, Pythagorean triples, Ramsey numbers and van der Waerden's problem. The proof of the theorem on Pythagorean triples was accomplished recently with the aid of a computer program based on mathematical logic. The chapter concludes with a digression on the ancient Babylonians' knowledge of Pythagorean triples.

C. Dudley Langford observed his son playing with colored blocks and noticed that he had laid them out in an interesting sequence. Chapter 9 presents his theorem on the conditions for such a sequence to be possible.

Chapter 10 contains the seven axioms of origami, together with the detailed calculations of the analytic geometry of the axioms, and characterizations of the folds as geometric loci.

Chapter 11 presents Eduard Lill's method and the origami fold proposed by Margharita P. Beloch. I introduce Lill's method as a magic trick so I won't spoil it by giving details here.

Chapter 12 shows that origami can perform constructions not possible with a straightedge and compass: trisecting an angle, squaring a circle and constructing a nonagon (a regular polygon with nine sides).

Chapter 13 presents the theorem by Georg Mohr and Lorenzo Mascheroni that any construction with a straightedge and compass can be performed using only a compass.

The corresponding claim that a straightedge only is sufficient is incorrect, because a straightedge cannot compute lengths that are square roots. Jean-Victor Poncelet conjectured and Jakob Steiner proved that a straightedge is sufficient, provided that there exists a single fixed circle somewhere in the plane (Chap. 14).

If two triangles have the same perimeter and the same area must they be congruent? That seems reasonable but it turns out not to be true, although it takes quite a bit of algebra and geometry to find a non-congruent pair as shown in Chap. 15.

Chapter 16 presents Gauss's tour-de-force: a proof that a heptadecagon (a regular polygon with seventeen sides) can be constructed using a straightedge and compass. By a clever argument on the symmetry of the roots of polynomials, he obtained a formula that uses only the four arithmetic operators and square roots. Gauss did not give an explicit construction of a heptadecagon, so the elegant construction by James Callagy is presented. The chapter concludes with constructions of a regular pentagon based on Gauss's method for the construction of a heptadecagon.

To keep the book as self-contained as possible, Appendix A collects proofs of theorems of geometry and trigonometry that may not be familiar to the reader.

## Style

- The reader is assumed to have a good knowledge of secondary-school mathematics, including:
  - Algebra: polynomials and division of polynomials, *monic* polynomials—those whose coefficient of the highest power is 1, quadratic equations, multiplication of expressions with exponents  $a^m \cdot a^n = a^{m+n}$ .
  - Euclidean geometry: congruent triangles  $\triangle ABC \cong \triangle DEF$  and the criteria for congruence, similar triangles  $\triangle ABC \sim \triangle DEF$  and the ratios of their sides, circles and their inscribed and central angles.
  - Analytic geometry: the cartesian plane, computing lengths and slopes of line segments, the formula for a circle.
  - Trigonometry: the functions sin, cos, tan and the conversions between them, angles in the unit circle, the trigonometric functions of angles reflected around an axis such as  $\cos(180^\circ - \theta) = -\cos \theta$ .
- Statements to be proved are called *theorems* with no attempt to distinguish between theorems, lemmas and corollaries.
- When a theorem follows a construction, the variables that appear in the theorem refer to labeled points, lines and angles in the figure accompanying the construction.
- The full names of mathematicians have been given without biographical information that can be found easily in Wikipedia.
- The book is written so that it is as self-contained as possible, but occasionally the presentation depends on advanced mathematical concepts and theorems that are given without proofs. In such cases, a summary of the material is presented in boxes which may be skipped.
- There are no exercises but the ambitious reader is invited to prove each theorem before reading the proof.

- Geometric constructions can be studied using software such as Geogebra.
- $\overline{AB}$  is used both for the name of a line segment and for the length of the segment.
- $\triangle ABC$  is used both for the name a triangle and for the area of the triangle.

## Acknowledgments

This book would never have been written without the encouragement of Abraham Arcavi who welcomed me to trespass on his turf of mathematics education. He also graciously wrote the foreword. Avital Elbaum Cohen and Ronit Ben-Bassat Levy were always willing to help me (re-)learn secondary-school mathematics. Oriah Ben-Lulu introduced me to the mathematics of origami and collaborated on the proofs. I am grateful to Michael Woltermann for permission to use several sections of his reworking of Heinrich Dörrie's book. Jason Cooper, Richard Krueel, Abraham Arcavi and the anonymous reviewers provided helpful comments.

I would like to thank the team at Springer for their support and professionalism, in particular the editor Richard Krueel.

The book is published under the Open Access program and I would like to thank the Weizmann Institute of Science for funding the publication.

The  $\LaTeX$  source files for the book (which include the  $TikZ$  source for the diagrams) are available at:

<https://github.com/motib/surprises>

*Mordechai (Moti) Ben-Ari*

# Contents

<b>1</b>	<b>The Collapsing Compass</b> .....	1
1.1	Construction with a Straightedge and Compass .....	2
1.2	Fixed Compasses and Collapsing Compasses .....	2
1.3	Euclid’s Construction for Copying a Line Segment .....	3
1.4	A Flawed Construction for Copying a Line Segment .....	5
1.5	Don’t Trust a Diagram .....	7
<b>2</b>	<b>Trisection of an Angle</b> .....	11
2.1	Approximate Trisections .....	11
2.2	Trisection Using a Neusis .....	16
2.3	Doubling the Cube with a Neusis .....	18
2.4	Trisection Using a Quadratrix .....	19
2.5	Constructible Numbers .....	21
2.6	Constructible Numbers As Roots of Polynomials .....	24
2.7	Impossibility of the Classical Constructions .....	26
<b>3</b>	<b>Squaring the Circle</b> .....	29
3.1	Kochański’s Construction .....	30
3.2	Ramanujan’s First Construction .....	32
3.3	Ramanujan’s Second Construction .....	35
3.4	Squaring a Circle Using a Quadratrix .....	38
<b>4</b>	<b>The Five-Color Theorem</b> .....	41
4.1	Planar Maps and Graphs .....	41
4.2	Euler’s Formula .....	44
4.3	Non-planar Graphs .....	45
4.4	The Degrees of the Vertices .....	46
4.5	The Six-Color Theorem .....	48
4.6	The Five-Color Theorem .....	48

4.7 Kempe’s Incorrect Proof of the Four-Color Theorem . . . . . 50

**5 How to Guard a Museum . . . . . 53**

5.1 Coloring Triangulated Polygons . . . . . 54

5.2 From Coloring of Polygons to Guarding a Museum . . . . . 56

5.3 Any Polygon Can Be Triangulated . . . . . 57

**6 Induction . . . . . 61**

6.1 The Axiom of Mathematical Induction . . . . . 61

6.2 Fibonacci Numbers . . . . . 63

6.3 Fermat Numbers . . . . . 66

6.4 McCarthy’s 91-function . . . . . 67

6.5 The Josephus Problem . . . . . 69

**7 Solving Quadratic Equations . . . . . 73**

7.1 Traditional Methods for Solving Quadratic Equations . . . . . 73

7.2 The Relation Between the Roots and the Coefficients . . . . . 74

7.3 Examples of Loh’s Method . . . . . 76

7.4 Derivation of the Traditional Formula . . . . . 78

7.5 Al-Khwarizmi’s Geometric Solution of Quadratic Equations . . . . . 78

7.6 Cardano’s Construction for Solving Cubic Equations . . . . . 79

7.7 They Weren’t Intimidated by Imaginary Numbers . . . . . 81

7.8 Lill’s Method and Carlyle’s Circle . . . . . 83

7.9 Numerical Computation of the Roots . . . . . 85

**8 Ramsey Theory . . . . . 89**

8.1 Schur triples . . . . . 89

8.2 Pythagorean Triples . . . . . 91

8.3 Van der Waerden’s problem . . . . . 92

8.4 Ramsey’s Theorem . . . . . 93

8.5 The Probabilistic Method . . . . . 94

8.6 SAT Solving . . . . . 96

8.7 Pythagorean Triples in Babylonian Mathematics . . . . . 100

**9 Langford’s Problem . . . . . 105**

9.1 Langford’s Problem as a Covering Problem . . . . . 106

9.2 For Which Values of  $N$  Is Langford’s Problem Solvable? . . . . . 107

9.3 Solution for  $L(4)$  . . . . . 110

- 10 The Axioms of Origami** ..... 113
  - 10.1 Axiom 1 ..... 114
  - 10.2 Axiom 2 ..... 115
  - 10.3 Axiom 3 ..... 116
  - 10.4 Axiom 4 ..... 118
  - 10.5 Axiom 5 ..... 119
  - 10.6 Axiom 6 ..... 121
  - 10.7 Axiom 7 ..... 128
  
- 11 Lill’s Method and the Beloch Fold** ..... 131
  - 11.1 A Magic Trick ..... 131
  - 11.2 Specification of Lill’s Method ..... 133
  - 11.3 Proof of Lill’s Method ..... 137
  - 11.4 The Beloch Fold ..... 138
  
- 12 Geometric Constructions Using Origami** ..... 141
  - 12.1 Abe’s Trisection of an Angle ..... 141
  - 12.2 Martin’s Trisection of an Angle ..... 143
  - 12.3 Messer’s Doubling of a Cube ..... 144
  - 12.4 Beloch’s Doubling of a Cube ..... 147
  - 12.5 Construction of a Regular Nonagon ..... 148
  
- 13 A Compass Is Sufficient** ..... 151
  - 13.1 What Is a Construction With Only a Compass? ..... 151
  - 13.2 Reflection of a Point ..... 152
  - 13.3 Construction of a Circle With a Given Radius ..... 153
  - 13.4 Addition and Subtraction of Line Segments ..... 154
  - 13.5 Construction of a Line Segment as a Ratio of Segments ..... 158
  - 13.6 Construction of the Intersection of Two Lines ..... 159
  - 13.7 Construction of the Intersection of a Line and a Circle ..... 161
  
- 14 A Straightedge and One Circle is Sufficient** ..... 163
  - 14.1 What Is a Construction With Only a Straightedge? ..... 163
  - 14.2 Construction of a Line Parallel to a Given Line ..... 164
  - 14.3 Construction of a Perpendicular to a Given Line ..... 167
  - 14.4 Copying a Line Segment in a Given Direction ..... 168
  - 14.5 Construction of a Line Segment as a Ratio of Segments ..... 169
  - 14.6 Construction of a Square Root ..... 169
  - 14.7 Construction of the Intersection of a Line and a Circle ..... 170
  - 14.8 Construction of the Intersection of Two Circles ..... 172

**15 Are Triangles with Equal Areas and Perimeters Congruent? . . . . . 175**

15.1 From a Triangle to an Elliptic Curve . . . . . 175

15.2 Solving the Equation for the Elliptic Curve . . . . . 178

15.3 Derivation of a Triangle From the Elliptic Curve . . . . . 180

**16 Construction of a Regular Heptadecagon . . . . . 183**

16.1 Construction of Regular Polygons . . . . . 184

16.2 The Fundamental Theorem of Algebra . . . . . 185

16.3 Roots of Unity . . . . . 185

16.4 Gauss’s Proof That a Heptadecagon Is Constructible . . . . . 187

16.5 Derivation of Gauss’s Formula . . . . . 191

16.6 Construction of a Heptadecagon . . . . . 192

16.7 Construction of a Regular Pentagon . . . . . 196

**A Theorems From Geometry and Trigonometry . . . . . 199**

A.1 Theorems About Triangles . . . . . 199

A.2 Trigonometric Identities . . . . . 202

A.3 The Angle Bisector Theorems . . . . . 211

A.4 Ptolemy’s Theorem . . . . . 212

A.5 Ceva’s Theorem . . . . . 215

A.6 Menelaus’s Theorem . . . . . 217

**References . . . . . 219**

**Index . . . . . 223**