

Springer Undergraduate Texts
in Mathematics and Technology

SUMAT

Heather A. Moon
Thomas J. Asaki
Marie A. Snipes

Application-Inspired Linear Algebra

 Springer

Springer Undergraduate Texts in Mathematics and Technology

Series Editors

Helge Holden, Department of Mathematical Sciences, Norwegian University of Science and Technology, Trondheim, Norway

Keri A. Kornelson, Department of Mathematics, University of Oklahoma, Norman, OK, USA

Editorial Board

Lisa Goldberg, Department of Statistics, University of California, Berkeley, Berkeley, CA, USA

Armin Iske, Department of Mathematics, University of Hamburg, Hamburg, Germany

Palle E.T. Jorgensen, Department of Mathematics, University of Iowa, Iowa City, IA, USA

Springer Undergraduate Texts in Mathematics and Technology (SUMAT) publishes textbooks aimed primarily at the undergraduate. Each text is designed principally for students who are considering careers either in the mathematical sciences or in technology-based areas such as engineering, finance, information technology and computer science, bioscience and medicine, optimization or industry. Texts aim to be accessible introductions to a wide range of core mathematical disciplines and their practical, real-world applications; and are fashioned both for course use and for independent study.

More information about this series at <https://link.springer.com/bookseries/7438>

Heather A. Moon · Thomas J. Asaki ·
Marie A. Snipes

Application-Inspired Linear Algebra

 Springer

Heather A. Moon
Department of Physics
and Astronomy
Washington State University
Pullman, WA, USA

Thomas J. Asaki
Department of Mathematics
and Statistics
Washington State University
Pullman, WA, USA

Marie A. Snipes
Department of Mathematics
and Statistics
Kenyon College
Gambier, OH, USA

ISSN 1867-5506 ISSN 1867-5514 (electronic)
Springer Undergraduate Texts in Mathematics and Technology
ISBN 978-3-030-86154-4 ISBN 978-3-030-86155-1 (eBook)
<https://doi.org/10.1007/978-3-030-86155-1>

Mathematics Subject Classification: 15-Axx

© Springer Nature Switzerland AG 2022

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To the curious learner, the creative thinkers, and to the explorers.

–Heather

*To the curious student, the motivated reader, the independent person in each of us,
and to the amazing people who inspire them.*

–Tom

To Ian

–Marie

Preface

We wrote this book with the aim of inspiring the reader to explore mathematics. Our goal is to provide opportunities for students to discover mathematical ideas *in the context of applications*. Before any formal mathematics, the text starts with two main data applications—radiography/tomography of images and heat diffusion—to inspire the creation and development of Linear Algebra concepts.

The applications are presented primarily through a sequence of explorations. Readers first learn about one aspect of a data application, and then, in an inquiry framework, they develop the mathematics necessary to investigate the application. After each exploration, the reader will see the standard definitions and theorems for a first-year Linear Algebra course, but with the added context of the applications.

A novel feature of this approach is that the applied problem *inspires* the mathematical theory, rather than the applied problem being presented after the relevant mathematics has been learned. Our goal is for students to organically experience the relevance and importance of the abstract ideas of linear algebra to real problems. We also want to give students a taste of research mathematics. Our explorations ask students to make conjectures and answer open-ended questions; we hope they demonstrate for students the living process of mathematical discovery.

Because of the application-inspired nature of the text, we created a path through introductory linear algebra material to naturally arise in the process of investigating two data applications. This led to a couple key content differences from many standard introductory linear algebra texts. First, we introduce vector spaces very early on as the appropriate settings for our problems. Second, we approach eigenvalue computations from an application/computation angle, offering a determinant-free method as well as the typical determinant method for calculating eigenvalues.

Although we have focused on two central applications that inspire the development of the linear algebra ideas in this text, there are a wide array of other applications and mathematical paths, many of which relate to data applications, that can be modeled with linear algebra. We have included “sign posts” for these applications and mathematical paths at moments where the reader has learned the necessary tools for exploring the application or mathematical path. These applications and mathematical paths are separated into three main areas: Data and Image Analysis (including Machine Learning), Dynamical Modeling, and Optimization and Optimal Design.

Outline of Text

In Chapter 1 we outline some of the fundamental ways that linear algebra is used in our world. We then introduce, with more depth, the applications of radiography/tomography, diffusion welding, and heat warping of images, which will frame our discussion of linear algebra concepts throughout the book.

Chapter 2 introduces systems of equations and vector spaces in the context of the applications. The chapter begins with an exploration (Section 2.1) of image data similar to what would be used for radiographs or the reconstructed images of brain slices. Motivated by a question about combining images, Section 2.2 outlines methods for solving systems of equations. (For more advanced courses, this chapter can be skipped.) In Section 2.3, we formalize properties of the set of images (images can be added, multiplied by scalars, etc.) and we use these properties to define a vector space. While Section 2.3 focuses on the vector spaces of images and Euclidean spaces, Section 2.4 introduces readers to a whole host of new vector spaces. Some of these (like polynomial spaces and matrix spaces) are standard, while other examples introduce vector spaces that arise in applications, including heat states, 7-bar LCD digits, and function spaces (including discretized function spaces). We conclude the chapter with a discussion of subspaces (Section 2.5), again motivated by the setting of images.

Chapter 3 delves into the fundamental ideas of linear combinations, span, and linear independence, and concludes with the development of bases and coordinate representations for vector spaces. Although the chapter does not contain any explorations, it is heavily motivated by explorations from the previous chapter. Specifically, the goal of determining if an image is an arithmetic combination of other images (from Section 2.1) drives the definition of linear combinations in Section 3.1, and also adds context to the abstract concepts of the span of a set of vectors (Section 3.2) and linear independence (Section 3.3). In Sections 3.4 and 3.5, we investigate how linearly independent spanning sets (bases) in the familiar spaces of images and heat states are useful for defining coordinates on those spaces. This allows us to match-up images and heat states with vectors in Euclidean spaces of the same dimension. We conclude the chapter with a “sign post” for regression analysis.

Chapter 4 covers linear transformations. In Section 4.1, readers are taken through an exploration of the radiographic transformation beginning with a definition of the transformation. Next, they use coordinates to represent this transformation with a matrix, and in Section 4.2, they investigate transformations more generally. In Section 4.3 readers see how the heat diffusion operator can be represented as a matrix, and in Section 4.4 they explore more generally how to represent arbitrary transformations between vector spaces by matrices of real numbers. In Section 4.6, the reader will return to the radiographic transformation and explore properties of the transformation, considering whether it is possible for two objects to produce the same radiograph, and whether there are any radiographs that are not produced by any objects. This exploration leads to the definitions of one-to-one and onto

linear transformations in Section 4.7. This is also where the critical idea of invertibility is introduced; in the radiographic transformation setting, if the transformation is invertible then reconstruction (tomography) is possible.

The goal of Chapter 5 is to understand invertibility so that we can solve inverse problems. In Section 5.1 readers consider what would happen if the radiographic transformation is not invertible. This leads to a study of subspaces related to the transformations (nullspace and range space). The section concludes with the rank-nullity theorem. In Section 5.2, the corresponding ideas for matrix representations of transformations (nullspace, row space, and column space) are discussed along with the introduction of the Invertible Matrix Theorem. In Section 5.3 the reader will reconstruct brain slice images for certain radiographic setups after developing the concept of a left inverse. We conclude this chapter with a “sign post” for linear programming.

Chapter 6 introduces eigenvector and eigenvalue concepts in preparation for simplifying iterative processes. Section 6.1 revisits the heat diffusion application. In this exploration, readers examine a variety of initial heat states, and observe that some heat states have a simple evolution while others do not. Combining this with the linearity of the diffusion operator leads to the idea of creating bases of these simple heat states. Section 6.2 formalizes the ideas of the previous heat diffusion exploration and introduces eigenvectors, eigenvalues, eigenspaces, and diagonalization. Using these constructs, in Section 6.3 readers, again, address the long-term behavior of various heat states, and start to make connections to other applications. We follow the application with Section 6.4 where we present many more applications described by repeated matrix multiplication or matrix/vector sequences. Within this chapter are “sign posts” for Fourier analysis, nonlinear optimization and optimal design, and for dynamical processes.

Chapter 7 includes the discussion on how to find suitable solutions to inverse problems when invertibility is not an option. In Section 7.1, motivated by the idea of determining the “degree of linear independence” of a set of images, readers will be introduced to the concepts of inner product and norm in a vector space. This chapter also develops the theory of orthogonality. Section 7.2 uses orthogonality to define orthogonal projections in Euclidean space along with general projections. The tools built here are then used to construct the Gram-Schmidt Process for producing an orthonormal basis for a vector space. In Section 7.3, motivated by ideas from earlier tomography explorations, we develop orthogonal transformations and related properties of symmetric matrices. Section 7.4 is an exploration in which the reader will learn about the concepts of maximal isomorphisms and pseudo-invertibility. In Section 7.5, readers will combine their knowledge about diagonalizable and symmetric transformations and orthogonality to more efficiently invert a larger class of radiographic transformations using singular value decomposition (SVD). The Final exploration, in Section 7.6, makes use of SVD to perform brain reconstructions. Readers will discover that SVD works well with clean data, but poorly for noisy data. At the end of this section, readers explore ideas for nullspace enhancements to reconstruct

brain images from noisy data. This final section is set up so that the reader can extend their knowledge of Linear Algebra in a grand finale exploration reaching into an active area of inverse problem research. Also included throughout Chapter 7 are “sign posts” for data analysis tools, including support vector machines, clustering, and principle component analysis.

Finally, Chapter 8 wraps up the text by describing the exploratory nature of applied mathematics and encourages the reader to continue using similar techniques on other problems.

Using This Text

The text is designed around a semester-long course. For a first course in Linear Algebra, we suggest including Chapters 1-6 with selected topics from Chapter 7 as time allows. Although the heat diffusion application is not fully resolved until Section 6.3 and the tomography application is not fully resolved until Section 7.6, one could reasonably conclude a 1-semester course after Section 6.2. At that point, some (relatively elementary) brain images have been reconstructed from radiographic data, a good exploration of Heat Diffusion has completed a study of Eigenvectors and Diagonalization, and tomography has motivated ideas that will lead to inner product, vector norm, projection, and the Gram-Schmidt Process. An outline from an example of our introductory courses is included on page xi.

Chapter 8 can be a great source of ideas for student projects. We encourage anyone using this text to consider applications discussed there.

This text has also been used for a more advanced second course in Linear Algebra. In this scenario, the instructor can move more rapidly through the first three chapters highlighting connections with the applications. The course could omit Sections 5.3 and 6.3 in order to have adequate time to complete the tomographic explorations in Chapter 7. Such a course could additionally include the derivation of the diffusion equation in Appendix B and/or a deeper understanding of radiographic transformations described in Appendix A.

Exercises

As mathematics is a subject best learned by doing, we have included exercises of a variety of types at many levels: concrete practice/computational, theoretical/proof-based, application-based, application-inspired/inquiry, and open-ended discussion exercises.

Computational Tools

One of the powerful aspects of Linear Algebra is its ability to solve large-scale problems arising in data analysis. We have designed our explorations to highlight this aspect of Linear Algebra. Many explorations include

Chapter	Sections	Title	# of (50-min) Classes
Ch 1		Introduction to Applications	1 Class
Ch 2		Vector Spaces	
	§2.1	Exploration: Digital Images	1 Class
	§2.2	Systems of Equations	2 Classes
	§2.3	Vector Spaces	1 Class
	§2.4	Vector Space Examples	1 Classes
	§2.5	Subspaces	2 Classes
Ch 3		Vector Space Arithmetic and Representations	
	§3.1	Linear Combinations	2 Classes
	§3.2	Span	2 Classes
	§3.3	Linear Independence	2 Classes
	§3.4	Bases	2 Classes
	§3.5	Coordinates	1 Class
Ch 4		Linear Transformations	
	§4.1	Exploration: Computing Radiographs	2 Classes
	§4.2	Linear Transformations	2 Classes
	§4.3	Exploration: Heat Diffusion	1 Class
	§4.4	Matrix Representations of Linear Transformations	
	§4.5	The Determinant of a Matrix	1 Class
	§4.6	Exploration: Tomography	1 Class
	§4.7	Transformation Properties (1-1 and Onto)	3 Classes
Ch 5		Invertibility	
	§5.1	Transformation Spaces	2 Classes
	§5.2	The Invertible Matrix Theorem	1 Class
	§5.3	Exploration: Tomography Without an Inverse	1 Class
Ch 6		Diagonalization	
	§6.1	Exploration: Heat State Evolution	1 Class
	§6.2	Eigenspaces	3 Classes
	§6.3	Exploration: Diffusion Welding	1 Class
	§6.4	Markov Processes	1 Class
Ch 7		Inner Product Spaces	
	§7.1	Inner Products	1 Class
	§7.2	Projections	1 Class

code for students to run in either MATLAB or the free, open-source software Octave. In most cases, the code can be run in online programming environments, eliminating the need for students to install software on their own computers.

Ancillary Materials

Readers using this text are invited to visit our website (www.imagemath.org) to access data sets and code for explorations. Instructors are able to create an account at our website so that they can download ancillary materials.

Materials available to instructors include all code and data sets for the explorations and instructor notes and expected solution paths for the explorations.

Pullman, USA
Pullman, USA
Gambier, USA

Heather A. Moon
Thomas J. Asaki
Marie A. Snipes

Acknowledgements

This book is an extension of the IMAGEMath collection of application-inspired modules, developed as a collaborative NSF project¹. We are grateful for the financial support from the NSF that made the IMAGEMath modules possible, and we are deeply grateful for the encouragement and mentorship of our program officer John Haddock for both the IMAGEMath and this follow-on book project.

We are grateful to the Park City Mathematics Institute's Undergraduate Faculty Program, where we met and laid the groundwork for our friendship, this work, and many more collaborations.

We extend our deepest gratitude to Dr. Amanda Harsy Ramsay for her enthusiasm for this work, early adoption of IMAGEMath materials, and invaluable feedback on the explorations in this text.

To Chris Camfield, thank you for being part of our first project.

We thank all of our linear algebra students for their willingness to look at mathematics from a different perspective, for insightful discussions through this material, for finding our typos, and for inspiring us to find better ways to ask you questions.

The genesis of this text was the body of notes HM wrote for her 2014 linear algebra class at St. Mary's College of Maryland weaving together IMAGEMath explorations with linear algebra concepts. We particularly thank students in this class for their formative comments and feedback.

We thank Kenyon College for its hospitality and financial support for several visits that HM and TA made to Gambier, Ohio, and we thank the Mathematics and Statistics Department for its support and encouragement for developing and using these materials at Kenyon College.

We thank Washington State University Mathematics Department for its support and encouragement for developing, disseminating, and using these materials.

We also thank Lane and Shirley Hathaway, at Palouse Divide Lodge, for providing a venue with such beautiful views and great walks.

To Ben Levitt, thank you for your interest in this work.

To the reviewers, thank you for your insightful feedback.

To the editors, thank you for all your help in preparing this work.

¹NSF-DUE 1503856, 1503929, 1504029, 1503870, and 1642095.

Contents

1	Introduction To Applications	1
1.1	A Sample of Linear Algebra in Our World.	1
1.1.1	Modeling Dynamical Processes	1
1.1.2	Signals and Data Analysis	2
1.1.3	Optimal Design and Decision-Making	2
1.2	Applications We Use to Build Linear Algebra Tools	3
1.2.1	CAT Scans	3
1.2.2	Diffusion Welding	4
1.2.3	Image Warping	5
1.3	Advice to Students	5
1.4	The Language of Linear Algebra	6
1.5	Rules of the Game	7
1.6	Software Tools	7
1.7	Exercises.	7
2	Vector Spaces.	9
2.1	Exploration: Digital Images	9
2.1.1	Exercises	11
2.2	Systems of Equations	14
2.2.1	Systems of Equations	14
2.2.2	Techniques for Solving Systems of Linear Equations.	18
2.2.3	Elementary Matrix.	29
2.2.4	The Geometry of Systems of Equations.	33
2.2.5	Exercises	35
2.3	Vector Spaces	38
2.3.1	Images and Image Arithmetic	38
2.3.2	Vectors and Vector Spaces	41
2.3.3	The Geometry of the Vector Space \mathbb{R}^3	48
2.3.4	Properties of Vector Spaces.	50
2.3.5	Exercises	52
2.4	Vector Space Examples	53
2.4.1	Diffusion Welding and Heat States	54
2.4.2	Function Spaces.	55
2.4.3	Matrix Spaces	58
2.4.4	Solution Spaces	59
2.4.5	Other Vector Spaces	61

2.4.6	Is My Set a Vector Space?	63
2.4.7	Exercises	64
2.5	Subspaces	66
2.5.1	Subsets and Subspaces	68
2.5.2	Examples of Subspaces	70
2.5.3	Subspaces of \mathbb{R}^n	73
2.5.4	Building New Subspaces	74
2.5.5	Exercises	77
3	Vector Space Arithmetic and Representations.	83
3.1	Linear Combinations	83
3.1.1	Linear Combinations	84
3.1.2	Matrix Products	88
3.1.3	The Matrix Equation $Ax = b$	92
3.1.4	The Matrix Equation $Ax = 0$	98
3.1.5	The Principle of Superposition	104
3.1.6	Exercises	106
3.2	Span	110
3.2.1	The Span of a Set of Vectors	111
3.2.2	To Span a Set of Vectors	114
3.2.3	Span X is a Vector Space	119
3.2.4	Exercises	122
3.3	Linear Dependence and Independence	124
3.3.1	Linear Dependence and Independence	126
3.3.2	Determining Linear (In)dependence	128
3.3.3	Summary of Linear Dependence	132
3.3.4	Exercises	133
3.4	Basis and Dimension	136
3.4.1	Efficient Heat State Descriptions	136
3.4.2	Basis	139
3.4.3	Constructing a Basis	142
3.4.4	Dimension	146
3.4.5	Properties of Bases	152
3.4.6	Exercises	157
3.5	Coordinate Spaces	160
3.5.1	Cataloging Heat States	160
3.5.2	Coordinates in \mathbb{R}^n	163
3.5.3	Example Coordinates of Abstract Vectors	165
3.5.4	Brain Scan Images and Coordinates	170
3.5.5	Exercises	171
4	Linear Transformations	175
4.1	Explorations: Computing Radiographs and the Radiographic Transformation	175
4.1.1	Radiography on Slices	176
4.1.2	Radiographic Scenarios and Notation	178
4.1.3	A First Example	179
4.1.4	Radiographic Setup Example	180
4.1.5	Exercises	180

4.2	Transformations	183
4.2.1	Transformations are Functions	185
4.2.2	Linear Transformations	185
4.2.3	Properties of Linear Transformations	192
4.2.4	Exercises	198
4.3	Explorations: Heat Diffusion	204
4.3.1	Heat States as Vectors	205
4.3.2	Heat Evolution Equation	208
4.3.3	Exercises	209
4.3.4	Extending the Exploration: Application to Image Warping	209
4.4	Matrix Representations of Linear Transformations	212
4.4.1	Matrix Transformations between Euclidean Spaces	212
4.4.2	Matrix Transformations	215
4.4.3	Change of Basis Matrix	224
4.4.4	Exercises	227
4.5	The Determinants of a Matrix	230
4.5.1	Determinant Calculations and Algebraic Properties	235
4.6	Explorations: Re-Evaluating Our Tomographic Goal	247
4.6.1	Seeking Tomographic Transformations	247
4.6.2	Exercises	248
4.7	Properties of Linear Transformations	250
4.7.1	One-To-One Transformations	250
4.7.2	Properties of One-To-One Linear Transformations	254
4.7.3	Onto Linear Transformations	258
4.7.4	Properties of Onto Linear Transformations	261
4.7.5	Summary of Properties	263
4.7.6	Bijections and Isomorphisms	263
4.7.7	Properties of Isomorphic Vector Spaces	264
4.7.8	Building and Recognizing Isomorphisms	267
4.7.9	Inverse Transformations	269
4.7.10	Left Inverse Transformations	273
4.7.11	Exercises	275
5	Invertibility	281
5.1	Transformation Spaces	282
5.1.1	The Nullspace	282
5.1.2	Domain and Range Spaces	287
5.1.3	One-to-One and Onto Revisited	292
5.1.4	The Rank-Nullity Theorem	295
5.1.5	Exercises	298
5.2	Matrix Spaces and the Invertible Matrix Theorem	301
5.2.1	Matrix Spaces	302
5.2.2	The Invertible Matrix Theorem	313
5.2.3	Exercises	322

- 5.3 Exploration: Reconstruction Without an Inverse 324
 - 5.3.1 Transpose of a Matrix 324
 - 5.3.2 Invertible Transformation. 325
 - 5.3.3 Application to a Small Example 325
 - 5.3.4 Application to Brain Reconstruction 326
- 6 Diagonalization 329**
 - 6.1 Exploration: Heat State Evolution. 330
 - 6.2 Eigenspaces and Diagonalizable Transformations 332
 - 6.2.1 Eigenvectors and Eigenvalues 333
 - 6.2.2 Computing Eigenvalues and Finding Eigenvectors 335
 - 6.2.3 Using Determinants to Find Eigenvalues. 344
 - 6.2.4 Eigenbases. 348
 - 6.2.5 Diagonalizable Transformations. 350
 - 6.2.6 Exercises 357
 - 6.3 Explorations: Long-Term Behavior and Diffusion
 - Welding Process Termination Criterion. 359
 - 6.3.1 Long-Term Behavior in Dynamical Systems 359
 - 6.3.2 Using MATLAB/OCTAVE to Calculate Eigenvalues and Eigenvectors 360
 - 6.3.3 Termination Criterion 362
 - 6.3.4 Reconstruct Heat State at Removal 364
 - 6.4 Markov Processes and Long-Term Behavior. 364
 - 6.4.1 Matrix Convergence 365
 - 6.4.2 Long-Term Behavior 369
 - 6.4.3 Markov Processes 372
 - 6.4.4 Exercises 374
- 7 Inner Product Spaces and Pseudo-Invertibility 379**
 - 7.1 Inner Products, Norms, and Coordinates. 379
 - 7.1.1 Inner Product. 381
 - 7.1.2 Vector Norm 385
 - 7.1.3 Properties of Inner Product Spaces 388
 - 7.1.4 Orthogonality. 390
 - 7.1.5 Inner Product and Coordinates. 395
 - 7.1.6 Exercises 398
 - 7.2 Projections 402
 - 7.2.1 Coordinate Projection 404
 - 7.2.2 Orthogonal Projection 409
 - 7.2.3 Gram-Schmidt Process. 422
 - 7.2.4 Exercises 429
 - 7.3 Orthogonal Transformations 432
 - 7.3.1 Orthogonal Matrices 433
 - 7.3.2 Orthogonal Diagonalization 439
 - 7.3.3 Completing the Invertible Matrix Theorem 442

7.3.4	Symmetric Diffusion Transformation	443
7.3.5	Exercises	445
7.4	Exploration: Pseudo-Inverting the Non-invertible	447
7.4.1	Maximal Isomorphism Theorem	447
7.4.2	Exploring the Nature of the Data Compression Transformation	449
7.4.3	Additional Exercises	451
7.5	Singular Value Decomposition	451
7.5.1	The Singular Value Decomposition	453
7.5.2	Computing the Pseudo-Inverse	465
7.5.3	Exercises	470
7.6	Explorations: Pseudo-Inverse Tomographic Reconstruction	471
7.6.1	The First Pseudo-Inverse Brain Reconstructions	471
7.6.2	Understanding the Effects of Noise.	473
7.6.3	A Better Pseudo-Inverse Reconstruction	473
7.6.4	Using Object-Prior Information	474
7.6.5	Additional Exercises	477
8	Conclusions	479
8.1	Radiography and Tomography Example	479
8.2	Diffusion.	480
8.3	Your Next Mathematical Steps	481
8.3.1	Modeling Dynamical Processes	482
8.3.2	Signals and Data Analysis	482
8.3.3	Optimal Design and Decision Making.	483
8.4	How to move forward.	484
8.5	Final Words	485
	Appendix A: Transmission Radiography and Tomography: A Simplified Overview	487
	Appendix B: The Diffusion Equation.	497
	Appendix C: Proof Techniques	501
	Appendix D: Fields	519
	Index	523