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Application-Inspired Linear Algebra

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To the curious learner, the creative thinkers, and to the explorers.

–Heather

To the curious student, the motivated reader, the independent person in each of us,
and to the amazing people who inspire them.

–Tom

To Ian

–Marie

Preface

We wrote this book with the aim of inspiring the reader to explore mathematics. Our goal is to provide opportunities for students to discover mathematical ideas *in the context of applications*. Before any formal mathematics, the text starts with two main data applications—radiography/tomography of images and heat diffusion—to inspire the creation and development of Linear Algebra concepts.

The applications are presented primarily through a sequence of explorations. Readers first learn about one aspect of a data application, and then, in an inquiry framework, they develop the mathematics necessary to investigate the application. After each exploration, the reader will see the standard definitions and theorems for a first-year Linear Algebra course, but with the added context of the applications.

A novel feature of this approach is that the applied problem *inspires* the mathematical theory, rather than the applied problem being presented after the relevant mathematics has been learned. Our goal is for students to organically experience the relevance and importance of the abstract ideas of linear algebra to real problems. We also want to give students a taste of research mathematics. Our explorations ask students to make conjectures and answer open-ended questions; we hope they demonstrate for students the living process of mathematical discovery.

Because of the application-inspired nature of the text, we created a path through introductory linear algebra material to naturally arise in the process of investigating two data applications. This led to a couple key content differences from many standard introductory linear algebra texts. First, we introduce vector spaces very early on as the appropriate settings for our problems. Second, we approach eigenvalue computations from an application/computation angle, offering a determinant-free method as well as the typical determinant method for calculating eigenvalues.

Although we have focused on two central applications that inspire the development of the linear algebra ideas in this text, there are a wide array of other applications and mathematical paths, many of which relate to data applications, that can be modeled with linear algebra. We have included "sign posts" for these applications and mathematical paths at moments where the reader has learned the necessary tools for exploring the application or mathematical path. These applications and mathematical paths are separated into three main areas: Data and Image Analysis (including Machine Learning), Dynamical Modeling, and Optimization and Optimal Design.

Outline of Text

In Chapter 1 we outline some of the fundamental ways that linear algebra is used in our world. We then introduce, with more depth, the applications of radiography/tomography, diffusion welding, and heat warping of images, which will frame our discussion of linear algebra concepts throughout the book.

Chapter 2 introduces systems of equations and vector spaces in the context of the applications. The chapter begins with an exploration (Section 2.1) of image data similar to what would be used for radiographs or the reconstructed images of brain slices. Motivated by a question about combining images, Section 2.2 outlines methods for solving systems of equations. (For more advanced courses, this chapter can be skipped.) In Section 2.3, we formalize properties of the set of images (images can be added, multiplied by scalars, etc.) and we use these properties to define a vector space. While Section 2.3 focuses on the vector spaces of images and Euclidean spaces, Section 2.4 introduces readers to a whole host of new vector spaces. Some of these (like polynomial spaces and matrix spaces) are standard, while other examples introduce vector spaces that arise in applications, including heat states, 7-bar LCD digits, and function spaces (including discretized function spaces). We conclude the chapter with a discussion of subspaces (Section 2.5), again motivated by the setting of images.

Chapter 3 delves into the fundamental ideas of linear combinations, span, and linear independence, and concludes with the development of bases and coordinate representations for vector spaces. Although the chapter does not contain any explorations, it is heavily motivated by explorations from the previous chapter. Specifically, the goal of determining if an image is an arithmetic combination of other images (from Section 2.1) drives the definition of linear combinations in Section 3.1, and also adds context to the abstract concepts of the span of a set of vectors (Section 3.2) and linear independence (Section 3.3). In Sections 3.4 and 3.5, we investigate how linearly independent spanning sets (bases) in the familiar spaces of images and heat states are useful for defining coordinates on those spaces. This allows us to match-up images and heat states with vectors in Euclidean spaces of the same dimension. We conclude the chapter with a "sign post" for regression analysis.

Chapter 4 covers linear transformations. In Section 4.1, readers are taken through an exploration of the radiographic transformation beginning with a definition of the transformation. Next, they use coordinates to represent this transformation with a matrix, and in Section 4.2, they investigate transformations more generally. In Section 4.3 readers see how the heat diffusion operator can be represented as a matrix, and in Section 4.4 they explore more generally how to represent arbitrary transformations between vector spaces by matrices of real numbers. In Section 4.6, the reader will return to the radiographic transformation and explore properties of the transformation, considering whether it is possible for two objects to produce the same radiograph, and whether there are any radiographs that are not produced by any objects. This exploration leads to the definitions of one-to-one and onto linear transformations in Section 4.7. This is also where the critical idea of invertibility is introduced; in the radiographic transformation setting, if the transformation is invertible then reconstruction (tomography) is possible.

The goal of Chapter 5 is to understand invertibility so that we can solve inverse problems. In Section 5.1 readers consider what would happen if the radiographic transformation is not invertible. This leads to a study of subspaces related to the transformations (nullspace and range space). The section concludes with the rank-nullity theorem. In Section 5.2, the corresponding ideas for matrix representations of transformations (nullspace, row space, and column space) are discussed along with the introduction of the Invertible Matrix Theorem. In Section 5.3 the reader will reconstruct brain slice images for certain radiographic setups after developing the concept of a left inverse. We conclude this chapter with a "sign post" for linear programming.

Chapter 6 introduces eigenvector and eigenvalue concepts in preparation for simplifying iterative processes. Section 6.1 revisits the heat diffusion application. In this exploration, readers examine a variety of initial heat states, and observe that some heat states have a simple evolution while others do not. Combining this with the linearity of the diffusion operator leads to the idea of creating bases of these simple heat states. Section 6.2 formalizes the ideas of the previous heat diffusion exploration and introduces eigenvectors, eigenvalues, eigenspaces, and diagonalization. Using these constructs, in Section 6.3 readers, again, address the long-term behavior of various heat states, and start to make connections to other applications. We follow the application with Section 6.4 where we present many more applications described by repeated matrix multiplication or matrix/vector sequences. Within this chapter are "sign posts" for Fourier analysis, nonlinear optimization and optimal design, and for dynamical processes.

Chapter 7 includes the discussion on how to find suitable solutions to inverse problems when invertibility is not an option. In Section 7.1, motivated by the idea of determining the "degree of linear independence" of a set of images, readers will be introduced to the concepts of inner product and norm in a vector space. This chapter also develops the theory of orthogonality. Section 7.2 uses orthogonality to define orthogonal projections in Euclidean space along with general projections. The tools built here are then used to construct the Gram-Schmidt Process for producing an orthonormal basis for a vector space. In Section 7.3, motivated by ideas from earlier tomography explorations, we develop orthogonal transformations and related properties of symmetric matrices. Section 7.4 is an exploration in which the reader will learn about the concepts of maximal isomorphisms and pseudo-invertibility. In Section 7.5, readers will combine their knowledge about diagonalizable and symmetric transformations and orthogonality to more efficiently invert a larger class of radiographic transformations using singular value decomposition (SVD). The Final exploration, in Section 7.6, makes use of SVD to perform brain reconstructions. Readers will discover that SVD works well with clean data, but poorly for noisy data. At the end of this section, readers explore ideas for nullspace enhancements to reconstruct brain images from noisy data. This final section is set up so that the reader can extend their knowledge of Linear Algebra in a grand finale exploration reaching into an active area of inverse problem research. Also included throughout Chapter 7 are "sign posts" for data analysis tools, including support vector machines, clustering, and principle component analysis.

Finally, Chapter 8 wraps up the text by describing the exploratory nature of applied mathematics and encourages the reader to continue using similar techniques on other problems.

Using This Text

The text is designed around a semester-long course. For a first course in Linear Algebra, we suggest including Chapters 1-6 with selected topics from Chapter 7 as time allows. Although the heat diffusion application is not fully resolved until Section 6.3 and the tomography application is not fully resolved until Section 7.6, one could reasonably conclude a 1-semester course after Section 6.2. At that point, some (relatively elementary) brain images have been reconstructed from radiographic data, a good exploration of Heat Diffusion has completed a study of Eigenvectors and Diagonalization, and tomography has motivated ideas that will lead to inner product, vector norm, projection, and the Gram-Schmidt Process. An outline from an example of our introductory courses is included on page xi.

Chapter 8 can be a great source of ideas for student projects. We encourage anyone using this text to consider applications discussed there.

This text has also been used for a more advanced second course in Linear Algebra. In this scenario, the instructor can move more rapidly through the first three chapters highlighting connections with the applications. The course could omit Sections 5.3 and 6.3 in order to have adequate time to complete the tomographic explorations in Chapter 7. Such a course could additionally include the derivation of the diffusion equation in Appendix B and/or a deeper understanding of radiographic transformations described in Appendix A.

Exercises

As mathematics is a subject best learned by doing, we have included exercises of a variety of types at many levels: concrete practice/computational, theoretical/proof-based, application-based, application-inspired/inquiry, and open-ended discussion exercises.

Computational Tools

One of the powerful aspects of Linear Algebra is its ability to solve large-scale problems arising in data analysis. We have designed our explorations to highlight this aspect of Linear Algebra. Many explorations include

Chapter	Sections	Title	# of (50-min) Classes
Ch 1		Introduction to Applications	1 Class
Ch 2		Vector Spaces	
	§2.1	Exploration: Digital Images	1 Class
	§2.2	Systems of Equations	2 Classes
	§2.3	Vector Spaces	1 Class
	§2.4	Vector Space Examples	1 Classes
	§2.5	Subspaces	2 Classes
Ch 3		Vector Space Arithmetic and Representations	
	§3.1	Linear Combinations	2 Classes
	§3.2	Span	2 Classes
	§3.3	Linear Independence	2 Classes
	§3.4	Bases	2 Classes
	§3.5	Coordinates	1 Class
Ch 4		Linear Transformations	
	§4.1	Exploration: Computing Radiographs	2 Classes
	§4.2	Linear Transformations	2 Classes
	§4.3	Exploration: Heat Diffusion	1 Class
	§4.4	Matrix Representations of Linear Transformations	
	§4.5	The Determinant of a Matrix	1 Class
	§4.6	Exploration: Tomography	1 Class
	§4.7	Transformation Properties (1-1 and Onto)	3 Classes
Ch 5		Invertibility	
	§5.1	Transformation Spaces	2 Classes
	§5.2	The Invertible Matrix Theorem	1 Class
	§5.3	Exploration: Tomography Without an Inverse	1 Class
Ch 6		Diagonalization	
	§6.1	Exploration: Heat State Evolution	1 Class
	§6.2	Eigenspaces	3 Classes
	§6.3	Exploration: Diffusion Welding	1 Class
	§6.4	Markov Processes	1 Class
Ch 7		Inner Product Spaces	
	§7.1	Inner Products	1 Class
	§7.2	Projections	1 Class

code for students to run in either MATLAB or the free, open-source software Octave. In most cases, the code can be run in online programming environments, eliminating the need for students to install software on their own computers.

Ancillary Materials

Readers using this text are invited to visit our website (www.imagemath.org) to access data sets and code for explorations. Instructors are able to create an account at our website so that they can download ancillary materials.

Materials available to instructors include all code and data sets for the explorations and instructor notes and expected solution paths for the explorations.

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