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Classical Lie Algebras at Infinity

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ISSN 1439-7382 ISSN 2196-9922 (electronic)
Springer Monographs in Mathematics
ISBN 978-3-030-89659-1 ISBN 978-3-030-89660-7 (eBook)
https://doi.org/10.1007/978-3-030-89660-7

Mathematics Subject Classification: 17B10, 17B55, 17B65

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This book originated from graduate topics courses given by the first author at Yale University and at the University of California, Berkeley. Since then, the exposition has grown to include some recent research results in the field. Our book is intended as a bridge between the traditional university graduate courses and research-level mathematics in representation theory. It is therefore appropriate for both an advanced graduate student entering the field and a research mathematician wishing to diversify and expand their knowledge.

The unifying theme of this book is the structure and representation theory of infinite-dimensional locally reductive Lie algebras and superalgebras. The first six chapters are foundational, while each of the last four chapters represents a research specialization in this large field, and for the most part can be studied independently.

A locally reductive Lie algebra is a direct limit of finite-dimensional reductive Lie algebras. This property is an essential difference from the well-studied Kac–Moody algebras of finite rank (see [K5]), which have finite-dimensional Cartan subalgebras but are not direct limits of finite-dimensional Lie algebras (unless they happen to be finite dimensional). An important class of locally reductive Lie algebras, namely root-reductive Lie algebras, can be considered as Kac–Moody algebras of infinite rank; however, they do not come with a choice of simple roots. Moreover, the representation theory of these Lie algebras exploits essentially the fact that they admit triangular decompositions that are not of Kac–Moody type.

An outline of the book is as follows. Chapter 1 is a fast-paced review of the structure theory of finite-dimensional Lie algebras, as needed for later chapters. In Chap. 2, we discuss finite-dimensional Lie superalgebras with more attention to detail for the unfamiliar reader. In Chap. 3, we introduce the central object of this book: root-reductive Lie algebras, which are defined as direct limits of finite-dimensional reductive Lie algebras with compatible root decompositions. In Chap. 4, we consider two generalizations, namely classically semisimple infinite rank Lie superalgebras and a class of direct limits of finite-dimensional reductive Lie algebras in which the root decompositions are no longer compatible.

Chapters 5 and 6 are devoted to the structure theory of the most basic root-reductive Lie algebras: $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$. More precisely, we describe Cartan subalgebras, Borel subalgebras, and parabolic subalgebras in maximal generality. A key notion for this is the notion of a generalized flag. This notion is specific to the Lie algebras we study and is not borrowed from the theory of classical finite-dimensional Lie algebras or Kac–Moody algebras.

In Chap. 7, we turn our attention to the representation theory of the Lie algebras $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$. The category of tensor modules is a very natural analog of the category of finite-dimensional representations of a classical Lie algebra (with the exception of spinor modules for $\mathfrak{o}(\infty)$). In contrast with the latter category, which is semisimple, tensor modules over $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$ form a non-semisimple Koszul category. We also briefly discuss tensor modules for Lie superalgebras.

In Chap. 8, we present some general results on weight modules. Since these results are not so widely known, even for finite-dimensional Lie algebras, we also discuss this case. For $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$, we pay special attention to recent results about simple bounded weight modules.

Chapter 9 is a brief review of generalized Harish-Chandra modules. These are modules with a locally finite action of a fixed subalgebra which, in contrast with the classical case considered by Harish-Chandra, does not have to be symmetric. Our main tools here are the Fernando–Kac subalgebra of a representation and the functor of cohomological induction, or the Zuckerman functor. Again, these results are not so widely known for finite-dimensional Lie algebras, and so we present them in some detail. In the case of $\mathfrak{sl}(\infty)$, $\mathfrak{o}(\infty)$, and $\mathfrak{sp}(\infty)$, the theory of generalized Harish-Chandra modules has only made its first steps.

Finally, in Chap. 10, we turn to geometry. We take up a jewel in classical representation theory, namely the Bott–Borel–Weil theorem, and discuss its known analogs for the Lie supergroups and locally reductive ind-groups. For convenience, we begin with presenting an explicit version of Demazure's proof for the Lie group $GL(V)$.

The necessary background for Chaps. 1–6 is just a standard introductory course on Lie algebras and their representations, while Chaps. 7 and 9 require some homological algebra, and Chap. 10 assumes basic knowledge of algebraic geometry and sheaf cohomology. Lie superalgebras appear only in Chaps. 2, 4, 7, and 10 and do not require prior knowledge beyond Lie algebras. Most chapters should fill several hour-long sessions if one decided to give a course based on the book. Exercises of various levels of difficulty are interlaced throughout the book to add depth to reading comprehension.

We see our exposition as an invitation to a wide open research area in the form of a guide, and our objective would be more than achieved if we have succeeded in confronting the reader with a thought-provoking complex mixture of mathematical ideas.

Acknowledgments

The first author thanks Yale University and the University of California at Berkeley where a first version of this book originated. Both authors thank the Weizmann Institute where this book was expanded and brought into its current form. The second author thanks Jacobs University, Bremen, for hosting while working on the book.

We express our gratitude to Alexander Baranov, Ivan Dimitrov, Karl-Hermann Neeb, and Gregg Zuckerman for useful discussions of the emerging invariants of locally simple locally finite Lie algebras, and to Herbert Gangl, Georgi Simidchiev, and Bogdan Vioreanu for help in the exploration of the open problems concerning supernatural numbers related to $\mathfrak{sl}(i, \mathrm{adj})$. In addition, we thank Kevin Coulembier, Elizabeth Dan-Cohen, Ivan Dimitrov, Dimitar Grantcharov, Siarhei Markouski, Todor Milev, Alexander Voronov, and Pablo Zadunaisky for reading drafts of the manuscript and making comments that helped improve the presentation. We are grateful to Ann Kostant for valuable guidance during the process of preparing this book for publication. Last but not least, we thank Mel de Veccio and Todor Milev for typing assistance.

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