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R. Michael Howe

An Invitation to Representation Theory

Polynomial Representations of the Symmetric Group

Springer

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ISSN 1615-2085 ISSN 2197-4144 (electronic)
Springer Undergraduate Mathematics Series
ISSN 2730-5813 ISSN 2730-5821 (electronic)
SUMS Readings
ISBN 978-3-030-98024-5 ISBN 978-3-030-98025-2 (eBook)
https://doi.org/10.1007/978-3-030-98025-2

Mathematics Subject Classification: 20-01, 20B05, 20B30, 20B35, 20Cxx, 20C05, 20C30

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

To my family, friends, colleagues, and students.

I'm only writing one book, so…

Preface

> A good stack of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one.
>
> *Paul Halmos*

Over the years I have had the opportunity to work with undergraduate mathematics students on various research and independent study projects, and I've found "representation theory of the symmetric group" to be an excellent vehicle to introduce them to more advanced mathematical concepts relatively early in their undergraduate careers. The basic idea is accessible to anyone with a solid background in linear algebra, and new content can be introduced as needed. Of course, it's also a beautiful subject in its own right that has many important applications.

The concept of the "symmetric group," that is, permutations acting on a set, along with the operation of composition and the existence of inverse permutations, is easy for the beginner to get their mind around, and serves as one of the first non-trivial examples of a non-Abelian group. Students are comfortable with polynomials, and the action of permutations on polynomials is straightforward and easy to understand.

Examples of such central concepts as "invariant subspaces," "irreducible subspaces," "isotypic subspaces," and "intertwining maps" can be explicitly constructed, and the further notions of "permutation representations" and "Young permutation modules" arise naturally. This setting provides examples that aid the intuition when moving on to more sophisticated territory, and much of the theory of the representations of so-called Lie groups is built on that of the symmetric group. On a somewhat historical note, some of the early study of the symmetric group and representation theory was by way of permuting the arguments of functions, called *substitutional analysis*. (See [R]).

There are LOTS of excellent books on the subject of representation theory, but they are essentially written at the graduate-student level (at least), and usually require familiarity with topics (topological spaces, tensor products, and multilinear algebra) beyond the experience of, say, a typical second-year mathematics undergraduate. The intended audience for this book will have completed a solid course in linear algebra, and so should be familiar with abstract vector spaces and the basic techniques of mathematical proofs. The reader may also be taking a modern (or "abstract") algebra course concurrently, but I assume no knowledge of modern algebra. Readers certainly don't need to know everything from undergraduate algebra, or even group theory, and I introduce the necessary concepts as the need arises. I recommend that the novice reader have a good linear algebra textbook, as well as a modern algebra textbook, at hand for reference. I caution that navigating multiple such references can be counter-productive, if for no other reason than differing notation. I also highly recommend working with a colleague and/or, if available, someone with more knowledge of the subject. Additionally, facility with a computer algebra system (CAS) such as Maple or Mathematica can ease some of the linear algebra computations.

Readers who are more mathematically advanced, but new to the subject, may find the discussion and the examples useful, but I emphasize that this book is written for novice (but motivated) mathematicians. Many of the explanations and proofs include more detail than would typically be included in a text for more "mathematically mature" readers, with the solutions to the exercises being particularly, if not annoyingly, detailed. Given a choice between saying too much or too little, I have erred on the side of saying too much. More advanced readers should be able to skim through the excessive details with minimal irritation.

I make no claim of originality, other than, perhaps, the level of the intended audience and the number of exercises. The material here is taken from a variety of sources, chosen and adapted so that it is accessible to novice mathematicians. I have tried to cite most of the sources where applicable, although I make no claim as to completeness of citation. I also make no attempt at efficiency. Some results and remarks are repeated as needed for clarity or precision, and for the benefit of the "grasshopper reader" who may jump in and around the text. The subject suffers from flourishing terminology, and I occasionally use terms interchangeably.

Throughout I endeavor to use the active voice. The term "we" can mean the author and the reader, as in "we see that…." Often the term "we" can be interpreted as "the mathematical community," as in "we show that …," since the ideas have been developed and exposed by dozens, if not hundreds, of mathematicians before us.

I would like to thank Chal Benson, Chris Davis, Carolyn Otto, and Gail Ratcliff (as well as the anonymous referees) for providing detailed and useful comments on early versions of this book. I would also like to express my appreciation to the scores of undergraduate students who participated in research and independent study projects with me, at least a dozen of whom have gone on to earn PhDs in mathematics.

Eau Claire, WI, USA R. Michael Howe

Introduction

At this introductory level, the theory of groups and their representations lies mostly within the mathematical realm of algebra, so let's first review what this means. An *algebraic structure* consists of an underlying set A, along with some ways of combining the elements of A to obtain another element in A by using *operations* that are required to satisfy certain *axioms*.

There are lots of algebraic structures: vector spaces, semigroups, rings, lattices; the list goes on. There are also other types of mathematical structures such as topological and analytic structures as well as hybrid structures such as topological groups and Hilbert spaces. The main concern of this book will be the algebraic structure of groups and modules. Groups are ubiquitous in mathematics and make precise the notion of symmetry.

Group representations are a type of module, and representation theory translates many group-theoretic problems into problems in the (usually easier) landscape of linear algebra. The theory has wide-ranging applications, from quantum mechanics [W] to voting theory [D] and to other areas of mathematics (algebraic geometry, invariant theory, multivariate statistics, combinatorics). It is the opinion of some science writers that "representation theory has served as a key ingredient in many of the most important discoveries in mathematics." [H].

One algebraic structure that readers are assumed to be familiar with is that of a *vector space*, with the operations of scalar multiplication and vector addition that satisfy the various associative, commutative, and distributive properties, etc. Some readers may wish to review the properties of vector spaces.

The scalars in a vector space are from a *field*, another algebraic structure with which readers should be familiar. The most common examples of fields are the rational numbers $\mathbb{Q}$, the real numbers $\mathbb{R}$, and the complex numbers $\mathbb{C}$, along with the operations of addition and multiplication that satisfy the various axioms of associativity, existence of identities and inverses, etc. Again, some readers may wish to review the properties of fields.

The *characteristic* of a field $\mathbb{F}$ is defined as the smallest natural number n such that[1]

$$na := \underbrace{a + a + \cdots + a}_{n \text{ summands}} = 0$$

for any $a \in \mathbb{F}$. If no such n exists, we say $\mathbb{F}$ has *characteristic zero*, as is the case with the familiar fields $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$. However, there are fields with positive characteristic (see Exercise 9.17), and the representation theory over such fields (so-called *modular representation theory*), even for uncomplicated groups, is still an active area of research. In this book, except for an occasional exercise, we only utilize fields of characteristic zero.

The most convenient field to use in most instances, and which we usually adopt, is the field of complex numbers $\mathbb{C}$, which is *algebraically closed*. That is, any polynomial equation with complex coefficients and positive degree has a solution in the complex numbers. In particular, this property guarantees the existence of eigenvalues of linear operators on a complex vector space, an attribute that is essential for some important results. Readers not fluent with complex numbers shouldn't panic; we will make only a few computations using them, and the moniker "complex," as well as "imaginary," is an unfortunate artifact of history.

Things start out pretty casually. In the first two chapters we develop the notions of a group, group actions, and group representations. Those who have completed a course in modern algebra should find this an easy read. Chapter 3 introduces some of the basic concepts and machinery of representation theory, and by the end of this chapter the reader should have a pretty good idea, supported by examples, of what "group representation theory" is about.

Chapter 4 examines the symmetric group in more detail. In Chap. 5, we give explicit decompositions of some polynomial spaces into irreducible representations, and Chap. 6 discusses two equivalent incarnations of the group algebra, a useful construction in representation theory and elsewhere.

The material now follows a more traditional exposition of the representation theory of the symmetric group, although we often relate it to representations on polynomial spaces. It also starts getting more technical, there is just no way to avoid it. You don't need to understand the proof of a theorem to apply the results, and novice readers can perhaps return later to understand the proofs and other technical details as their knowledge and abilities increase. Of course, it would be mathematical heresy to not include the proofs, and many of them are presented in a series of exercises—with solutions—that outline the basic ideas.

In Chap. 7 we discuss characters, a kind of function that "characterizes" a representation, and use these to label all of the irreducible representations of the symmetric group. Group characters are a useful tool in the study of group representations, and every book on representation theory should include a discussion

[1] The symbol := commonly denotes a definition.

of them, but characters are not essential to the more "constructive" approach that is our focus. In Chap. 8, we actually produce "realizations" of these representations in the group algebra and in polynomial spaces.

In Chap. 9 we introduce cosets, a basic construction in algebra, which are essential in order to construct representations of a group that are "induced" from that of a subgroup. Here, again, we present two (of several) equivalent versions of induced representations.

Chapter 10 describes direct products of groups, another standard algebraic construction, of which Young subgroups are an example. The trivial representations of Young subgroups are "induced" to representations called Young permutation modules. We also construct a new vector space of "polytabloids" that carries a representation of the symmetric group, and that has theoretical and computational advantages. We then relate these representations in polytabloid spaces to representations in polynomial spaces.

In Chap. 11, we construct the unique irreducible representation that appears in each Young permutation module, called a Specht module. Chapter 12 decomposes Young permutation modules into their irreducible components. Finally, in Chap. 13, we examine how the irreducible representations of S_n decompose when restricted to S_{n-1} or induced to S_{n+1}.

* An asterisk following an Exercise, Remark, etc., indicates that there is further information in the "Hints and Additional Comments" section at the end of each chapter. Of course, you should first try to work through them on your own.

Along the way, we will occasionally stray into related areas of mathematics that are suggested by our investigations. These are usually asterisked remarks placed at the end of the chapter so as not to break the logical flow of the main subject matter.

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