Lecture Notes in Mathematics 2319

Rolf Schneider

Convex Cones

Geometry and Probability

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Preface

In order to guide the reader into these notes, it seems appropriate to recall a word from the title: ‘Geometry’. This emphasizes our viewpoint, and it indicates that we will not deal with the important role of convex cones in such fields as ordered vector spaces (e.g., [3]), measure theory (e.g., [25]), homogeneous or symmetric cones (e.g., [56]), or conic optimization. Rather, we concentrate on intuitive and elementary geometric appearances of convex cones, for example in the investigation of polyhedra and in stochastic geometry.

Besides introducing the reader to the fundamental facts about convex cones and their geometric functionals, this is a selection, illuminating different aspects of the geometry of convex cones. The principles guiding our choices are twofold. Some older or newer results about convex cones or their applications we found so remarkable that we think they should be pointed out, preserved also outside their original sources, and presented to a wider public. Other topics had the advantage that we were more familiar with them, after studying them in detail, and they were included at appropriate places. Accordingly, much of the material is close to various research articles, and the selection is rather subjective.

In the previous decade, some questions from applied mathematics, concerning, for instance, the average case analysis of conic optimization problems, or demixing by convex optimization under a probabilistic model, were treated in a way leading to increased interest in non-trivial intersections of convex cones. As an example, the following question is posed in [11, p. 227]: *“What is the probability that a randomly rotated convex cone shares a ray with a fixed cone?”* If ‘randomly’ is interpreted to imply uniform distribution, the probability in question can be calculated via the kinematic formula of conic (or, equivalently, spherical) integral geometry. While the spherical, and thus the conic, kinematic formula was already known, the new applications required additional information about the functionals appearing in it, the conic intrinsic volumes. These can be considered as the conic counterparts to the intrinsic volumes of convex bodies. The investigations of the new applications of the conic intrinsic volumes, of which we mention [5, 8, 9, 11, 127, 128, 129], required refined information, such as the explicit values of the conic intrinsic volumes for special cones or estimates of their asymptotic behavior in high dimensions. The new interest in conic intrinsic volumes and the conic kinematic formula was also a motivation for developing new and simplified approaches to known results, as in [6] and [10].

Another incentive for collecting observations about convex cones came from different publications on random cones, or on intersections of cones with random subspaces. The early paper [50] treated random cones generated by tessellations of $\mathbb{R}^d$ by independent random hyperplanes through the origin, or as the positive hull of independent random vectors, with quite general assumptions on the distributions. The article [52] investigated, among other things, random linear images of the nonnegative orthant in a higher-dimensional space. On the other hand, beautiful probabilistic applications were made of intersections of random linear subspaces with the cones of special conical tessellations; see [113, 114]. More recently, various different aspects of random cones were studied intensively, in [66, 67, 68, 69, 108, 109, 110, 111]. Reading such articles turned out to be a stimulus to have also a look at older publications about convex cones.

The use of convex cones is classical, of course, in the geometry of polyhedra. Normal cones and tangent cones are indispensable and familiar for anyone working with polyhedra. But also some rarer appearances of cones in the geometry of polyhedra deserve interest and should perhaps be more widely known. When reading new and older articles on convex cones, the idea arose of collecting various geometric facts on cones and of presenting them coherently. The selection of topics was, as mentioned, a matter of personal interests, though guided by the hope that greater diversity might help more readers to find something of interest for them.

Chapter 1 lays the foundations, mainly on convex cones and polyhedra. It collects some special results which are needed later. Particular emphasis is on valuations and, connected to this, on identities for characteristic functions. Polarity of convex cones is studied from various viewpoints.

Chapter 2 introduces the basic valuations that are used to measure polyhedral cones: the conic intrinsic volumes and the Grassmann angles (or conic quermassintegrals). It establishes relations for them and between them, making use of identities for characteristic functions and of a first integral-geometric formula. Valuations on polyhedral cones are then used to establish Gauss–Bonnet type theorems and tube formulas for compact general polyhedra.

A cone yields by intersection a subset of the unit sphere, and a subset of the unit sphere uniquely determines a cone. Therefore, the geometry of cones is equivalent to the geometry of subsets of the sphere. Treating cones in Euclidean space, where we can use the linear structure, has several advantages and makes the presentation easier. Sometimes, however, spherical geometry is needed, or is more appropriate. Chapter 3 treats, therefore, relations to spherical geometry. It provides some calculations for later applications, and also discusses some spherical inequalities, which can be re-interpreted in the geometry of cones.

Chapter 4 deals with substantial metric properties of and manipulations with convex cones. The 'Master Steiner formula' of McCoy and Tropp is proved, in a generalized local form, involving the local versions of the conic intrinsic volumes, the conic support measures. This is first done for polyhedral cones and then extended, by continuity, to general convex cones. As an outcome, the conic intrinsic volumes are thus defined as continuous functionals on general closed convex cones. Then the kinematic formula of integral geometry is proved for curvature measures of convex cones. Its global form provides the probability of the event that a uniform random cone has a common ray with a fixed cone. A concentration property of the conic intrinsic volumes around the statistical dimension then leads to a threshold phenomenon. The chapter deals briefly with inequalities for conic intrinsic volumes, and the weak continuity of the conic support measures is strengthened to Hölder continuity with respect to suitable metrics.

Finitely many hyperplanes through the origin generate a tessellation of the space into polyhedral convex cones. Chapter 5 treats these cones, in several different ways. First, a formula is proved giving the sum of the kth conic intrinsic volumes of the j-faces of such a tessellation. It reveals that this sum depends only on the combinatorics of the central arrangement provided by the hyperplanes. Special advantage is drawn from this fact in the determination of an absorption probability concerning a certain random walk. The rest of the chapter deals with the situation when the given hyperplanes are random, with a distribution satisfying some mild assumptions. They then give rise to different models of random cones, and for these, the expectations of various geometric functionals are determined. Similar results are obtained for lower-dimensional faces of the random tessellation. The final section is concerned with probabilities of non-trivial intersections for isotropic random cones.

Chapter 6 continues the investigation of random cones, under various different aspects. A simple way to generate a random cone is to take the image of a fixed cone under a random linear map. The first two sections deal with such random cones. The behavior of random cones in high dimensions is the topic of the next three sections. Random cones in halfspaces are briefly considered in the last section.

The role that convex cones play in Chapter 7 is quite different. Here a given convex cone serves as a cage for a convex hypersurface which is asymptotic to the cone. Examples appear in an old conjecture of Calabi, according to which every complete hyperbolic affine hypersphere is asymptotic to the boundary of a convex cone with apex at the center, and that every pointed convex cone with interior points determines a one-parameter family of affine hyperbolic hyperspheres which are asymptotic to the boundary of the cone. With a distinctly different motivation, Khovanskiĭ and Timorin [115] were led to consider convex sets K contained in a fixed cone C such that $C \setminus K$ is bounded. More generally, we shall study C-coconvex sets, by which we understand sets of the form $C \setminus K$, where C is a pointed closed convex cone with interior points, $K \subseteq C$ is a closed convex set, and $C \setminus K$ has finite volume. For such sets, we shall develop the first steps of a Brunn–Minkowski theory, relating volume and a kind of addition.

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Contents