

Volume 8

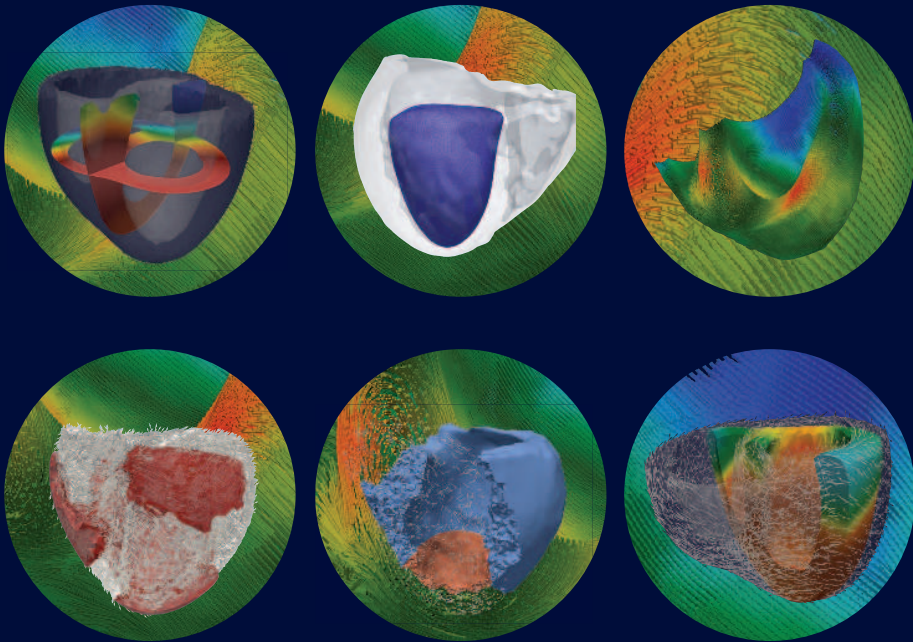
Numerical Models for Differential Problems

Second Edition

Alfio Quarteroni

MS&A

Modeling, Simulation & Applications



MS&A

Volume 8

Editor-in-Chief

A. Quarteroni

Series Editors

T. Hou

C. Le Bris

A.T. Patera

E. Zuazua

For further volumes:

<http://www.springer.com/series/8377>

Alfio Quarteroni

Numerical Models for Differential Problems

Second Edition

 Springer

Alfio Quarteroni
CMCS-MATHICSE
Ecole Polytechnique Fédérale de Lausanne
Switzerland
and
MOX, Department of Mathematics “F. Brioschi”
Politecnico di Milano
Italy

Translated by Silvia Quarteroni from the original Italian edition:
A. Quarteroni, *Modellistica Numerica per Problemi Differenziali*. 5^a ed.,
Springer-Verlag Italia, Milano 2012
Linguistic copy-editing: Simon Chiossi

ISSN print edition: 2037-5255 ISSN electronic edition: 2037-5263
MS&A – Modeling, Simulation & Applications
ISBN 978-88-470-5521-6 ISBN 978-88-470-5522-3 (eBook)
DOI 10.1007/978-88-470-5522-3
Springer Milan Heidelberg New York Dordrecht London

Library of Congress Control Number: 2013945787

© Springer-Verlag Italia 2014

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed. Exempted from this legal reservation are brief excerpts in connection with reviews or scholarly analysis or material supplied specifically for the purpose of being entered and executed on a computer system, for exclusive use by the purchaser of the work. Duplication of this publication or parts thereof is permitted only under the provisions of the Copyright Law of the Publisher's location, in its current version, and permission for use must always be obtained from Springer. Permissions for use may be obtained through RightsLink at the Copyright Clearance Center. Violations are liable to prosecution under the respective Copyright Law. The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

While the advice and information in this book are believed to be true and accurate at the date of publication, neither the authors nor the editors nor the publisher can accept any legal responsibility for any errors or omissions that may be made. The publisher makes no warranty, express or implied, with respect to the material contained herein.

9 8 7 6 5 4 3 2 1

Cover-Design: Beatrice , Milano

The cardiac numerical simulations reported on the front cover are due to Ricardo Ruiz Baier from CMCS-MATHICSE, EPFL, Lausanne
Typesetting with \LaTeX : PTP-Berlin, Protago \TeX -Production GmbH, Germany (www.ptp-berlin.de)

Springer is a part of Springer Science+Business Media (www.springer.com)

To Fulvia, Silvia and Marzia

Preface to the second edition

Differential equations (DEs) are the foundation on which many mathematical models for real-life applications are built. These equations can seldom be solved in ‘closed’ form: in fact, the exact solution can rarely be characterized through explicit, and easily computable, mathematical formulae. Almost invariably one has to resort to appropriate numerical methods, whose scope is the approximation (or discretization) of the exact differential model and, hence, of the exact solution.

This is the second edition of a book that first appeared in 2009. It presents in a comprehensive and self-contained way some of the most successful numerical methods for handling DEs, for their analysis and their application to classes of problems that typically show up in the applications.

Although we mostly deal with partial differential equations (PDEs), both for steady problems (in multiple space dimensions) and time-dependent problems (with one or several space variables), part of the material is specifically devoted to ordinary differential equations (ODEs) for one-dimensional boundary-value problems, especially when the discussion is interesting in itself or relevant to the PDE case.

The primary concern is on the finite-element (FE) method, which is the most popular discretization technique for engineering design and analysis. We also address other techniques, albeit to a lesser extent, such as finite differences (FD), finite volumes (FV), and spectral methods, including further *ad-hoc* methods for specific types of problems. The comparative assessment of the performance of different methods is discussed, especially when it sheds light on their mutual interplay.

We also introduce and analyze numerical strategies aimed at reducing the computational complexity of differential problems: these include operator-splitting and fractional-step methods for time discretization, preconditioning, techniques for grid adaptivity, domain decomposition (DD) methods for parallel computing, and reduced-basis (RB) methods for solving parametrized PDEs efficiently.

Besides the classical elliptic, parabolic and hyperbolic linear equations, we treat more involved model problems that arise in a host of applicative fields: linear and nonlinear conservation laws, advection-diffusion equations with dominating advection, Navier-Stokes equations, saddle-point problems and optimal-control problems.

Here is the contents' summary of the various chapters.

Chapter 1 briefly surveys PDEs and their classification, while Chapter 2 introduces the main notions and theoretical results of functional analysis that are extensively used throughout the book.

In Chapter 3 we illustrate boundary-value problems for elliptic equations (in one and several dimensions), present their weak or variational formulation, treat boundary conditions and analyze well-posedness. Several examples of physical interest are introduced.

The book's first cornerstone is Chapter 4, where we formulate Galerkin's method for the numerical discretization of elliptic boundary-value problems and analyze it in an abstract functional setting. We then introduce the Galerkin FE method, first in one dimension, for the reader's convenience, and then in several dimensions. We construct FE spaces and FE interpolation operators, prove stability and convergence results and derive several kinds of error estimates. Eventually, we present grid-adaptive procedures based on either *a priori* or *a posteriori* error estimates.

The numerical approximation of parabolic problems is explained in Chapter 5: we begin with semi-discrete (continuous in time) Galerkin approximations, and then consider fully-discrete approximations based on FD schemes for time discretization. For both approaches stability and convergence are proven.

Chapters 6, 7 and 8 are devoted to the algorithmic features and the practical implementation of FE methods. More specifically, Chapter 6 illustrates the main techniques for grid generation, Chapter 7 surveys the basic algorithms for the solution of ill-conditioned linear algebraic systems that arise from the approximation of PDEs, and Chapter 8 presents the main operational phases of a FE code, together with a complete working example.

The basic principles underlying finite-volume methods for the approximation of diffusion-transport-reaction equations are discussed in Chapter 9. FV methods are commonly used in computational fluid dynamics owing to their intrinsic, built-in conservation properties.

Chapter 10 addresses the multi-faceted aspects of spectral methods (Galerkin, collocation, and the spectral-element method), analyzing thoroughly the reasons for their superior accuracy properties.

Galerkin discretization techniques relying on discontinuous polynomial subspaces are the subject of Chapter 11. We present, more specifically, the discontinuous Galerkin (DG) method and the mortar method, together with their use in the context of finite elements or spectral elements.

Chapter 12 focuses on singularly perturbed elliptic boundary-value problems, in particular diffusion-transport equations and diffusion-reaction equations, with small diffusion. The exact solutions to this type of problems can exhibit steep gradients in tiny subregions of the computational domains, the so-called internal or boundary layers. A great deal of attention is paid to stabilization techniques meant to prevent the on-rise of oscillatory numerical solutions. Upwinding techniques are discussed for FD approximations, and their analogy with FE with artificial diffusion is analyzed. We introduce and discuss other stabilization approaches in the FE context, as well, which

lead to the sub-grid generalized Galerkin methods, the Petrov-Galerkin methods and Galerkin's Least-Squares method.

The ensuing three chapters form a thematic unit focusing on the approximation of first-order hyperbolic equations. Chapter 13 addresses classical FD methods. Stability is investigated using both the energy method and the Von Neumann analysis. Using the latter we also analyze the properties of dissipation and dispersion featured by a numerical scheme. Chapter 14 is devoted to spatial approximation by FE methods, including the DG methods and spectral methods. Special emphasis is put on characteristic compatibility conditions for the boundary treatment of hyperbolic systems. A very quick overview of the numerical approximation of nonlinear conservation laws is found in Chapter 15. Due to the relevance of this particular topic the interested reader is advised to consult the specific monographs mentioned in the references.

In Chapter 16 we discuss the Navier-Stokes equations for incompressible flows, plus their numerical approximation by FE, FV and spectral methods. A general stability and convergence theory is developed for spatial approximation of saddle-point problems, which comprises strategies for stabilization. Next we propose and analyze a number of time-discretization approaches, among which finite differences, characteristic methods, fractional-step methods and algebraic factorization techniques. Special attention is devoted to the numerical treatment of interfaces in the case of multi-phase flows.

Chapter 17 discusses the issue of optimal control for elliptic PDEs. The problem is first formulated at the continuous level, where conditions of optimality are obtained using two different methods. Then we address the interplay between optimization and numerical approximation. We present several examples, some of them elementary in character, others involving physical processes of applicative relevance.

Chapter 18 regards domain-decomposition methods. These techniques are specifically devised for parallel computing and for the treatment of multiphysics' PDE problems. The families of Schwarz methods (with overlapping subdomains) and Schur methods (with disjoint subdomains) are illustrated, and their convergence properties of optimality (grid invariance) and scalability (subdomain-size invariance) studied. Several examples of domain-decomposition preconditioners are provided and tested numerically.

Finally, in Chapter 19 we introduce the reduced-basis (RB) method for the efficient solution of PDEs. RB methods allow for the rapid and reliable evaluation of input/output relationships in which the output is expressed as a functional of a field variable that is the solution of a parametrized PDE. Parametrized PDEs model several processes relevant in applications such as steady and unsteady transfer of heat or mass, acoustics, solid and fluid mechanics, to mention a few. The input-parameter vector variously characterizes the geometric configuration of the domain, physical properties, boundary conditions or source terms. The combination with an efficient *a posteriori* error estimate, and the splitting between offline and online calculations, are key factors for RB methods to be computationally successful.

Many important topics that would have deserved a proper treatment were touched only partially (in some cases completely ignored). This depends on the desire to offer a reasonably-sized textbook on one side, and our own experience on the other. The

list of notable omissions includes, for instance, the approximation of equations for the structural analysis and the propagation of electromagnetic waves. Detailed studies can be found in the references' specialized literature.

This text is intended primarily for graduate students in Mathematics, Engineering, Physics and Computer Science and, more generally, for computational scientists. Each chapter is meant to provide a coherent teaching unit on a specific subject. The first eight chapters, in particular, should be regarded as a comprehensive and self-contained treatise on finite elements for elliptic and parabolic PDEs. Chapters 9–16 represent an advanced course on numerical methods for PDEs, while the last three chapters contain more subtle and sophisticated topics for the numerical solution of complex PDE problems.

This work has been used as a textbook for graduate-level courses at the Politecnico di Milano and the École Polytechnique Fédérale de Lausanne. We would like to thank the many people – students, colleagues and readers – who contributed, at various stages and in many different ways, to its preparation and to the improvement of early drafts. A (far from complete) list includes Paola Antonietti, Luca Dedé, Marco Discacciati, Luca Formaggia, Loredana Gaudio, Paola Gervasio, Andrea Manzoni, Stefano Micheletti, Nicola Parolini, Anthony T. Patera, Luca Pavarino, Simona Perotto, Gianluigi Rozza, Fausto Saleri, Benjamin Stamm, Alberto Valli, Alessandro Veneziani, and Cristoph Winkelmann. Special thanks go to Luca Paglieri for the technical assistance, to Francesca Bonadei of Springer for supporting this project since its very first Italian edition, and, last but not least, to Silvia Quarteroni for the translation from Italian and to Simon G. Chiossi for the linguistic revision of the second edition.

Milan and Lausanne, October 2013

Alfio Quarteroni

Contents

1	A brief survey of partial differential equations	1
1.1	Definitions and examples	1
1.2	Numerical solution	3
1.3	PDE Classification	5
1.3.1	Quadratic form associated to a PDE	8
1.4	Exercises	9
2	Elements of functional analysis	11
2.1	Functionals and bilinear forms	11
2.2	Differentiation in linear spaces	13
2.3	Elements of distributions	15
2.3.1	Square-integrable functions	17
2.3.2	Differentiation in the sense of distributions	18
2.4	Sobolev spaces	20
2.4.1	Regularity of the spaces $H^k(\Omega)$	21
2.4.2	The space $H_0^1(\Omega)$	22
2.4.3	Trace operators	23
2.5	The spaces $L^\infty(\Omega)$ and $L^p(\Omega)$, with $1 \leq p < \infty$	24
2.6	Adjoint operators of a linear operator	26
2.7	Spaces of time-dependent functions	27
2.8	Exercises	29
3	Elliptic equations	31
3.1	An elliptic problem example: the Poisson equation	31
3.2	The Poisson problem in the one-dimensional case	32
3.2.1	Homogeneous Dirichlet problem	33
3.2.2	Non-homogeneous Dirichlet problem	39
3.2.3	Neumann Problem	39
3.2.4	Mixed homogeneous problem	40
3.2.5	Mixed (or Robin) boundary conditions	40
3.3	The Poisson problem in the two-dimensional case	41

3.3.1	The homogeneous Dirichlet problem	41
3.3.2	Equivalence, in the sense of distributions, between weak and strong form of the Dirichlet problem	43
3.3.3	The problem with mixed, non homogeneous conditions	44
3.3.4	Equivalence, in the sense of distributions, between weak and strong form of the Neumann problem	47
3.4	More general elliptic problems	48
3.4.1	Existence and uniqueness theorem	50
3.5	Adjoint operator and adjoint problem	51
3.5.1	The nonlinear case	55
3.6	Exercises	56
4	The Galerkin finite element method for elliptic problems	61
4.1	Approximation via the Galerkin method	61
4.2	Analysis of the Galerkin method	63
4.2.1	Existence and uniqueness	63
4.2.2	Stability	64
4.2.3	Convergence	64
4.3	The finite element method in the one-dimensional case	67
4.3.1	The space X_h^1	67
4.3.2	The space X_h^2	69
4.3.3	The approximation with linear finite elements	71
4.3.4	Interpolation operator and interpolation error	73
4.3.5	Estimate of the finite element error in the H^1	75
4.4	Finite elements, simplices and barycentric coordinates	76
4.4.1	An abstract definition of finite element in the Lagrangian case	76
4.4.2	Simplexes	78
4.4.3	Barycentric coordinates	78
4.5	The finite element method in the multi-dimensional case	80
4.5.1	Finite element solution of the Poisson problem	82
4.5.2	Conditioning of the stiffness matrix	85
4.5.3	Estimate of the approximation error in the energy norm	88
4.5.4	Estimate of the approximation error in the L^2 norm	95
4.6	Grid adaptivity	98
4.6.1	A priori adaptivity based on derivatives reconstruction	100
4.6.2	A posteriori adaptivity	103
4.6.3	Numerical examples of adaptivity	107
4.6.4	A posteriori error estimates in the L^2 norm	111
4.6.5	A posteriori estimates of a functional of the error	112
4.7	Exercises	114

5	Parabolic equations	121
5.1	Weak formulation and its approximation	122
5.2	A priori estimates	125
5.3	Convergence analysis of the semi-discrete problem	128
5.4	Stability analysis of the θ -method	132
5.5	Convergence analysis of the θ -method	135
5.6	Exercises	138
6	Generation of 1D and 2D grids	141
6.1	Grid generation in 1D	141
6.2	Grid of a polygonal domain	144
6.3	Generation of structured grids	146
6.4	Generation of non-structured grids	149
6.4.1	Delaunay triangulation	149
6.4.2	Advancing front technique	153
6.5	Regularization techniques	155
6.5.1	Diagonal swap	156
6.5.2	Node displacement	157
7	Algorithms for the solution of linear systems	161
7.1	Direct methods	161
7.2	Iterative methods	164
7.2.1	Classical iterative methods	164
7.2.2	Gradient and conjugate gradient methods	167
7.2.3	Krylov subspace methods	169
7.2.4	The Multigrid method	175
8	Elements of finite element programming	179
8.1	Working steps of a finite element code	179
8.1.1	The code in a nutshell	182
8.2	Numerical computation of integrals	183
8.2.1	Numerical integration using barycentric coordinates	185
8.3	Storage of sparse matrices	187
8.4	Assembly	189
8.4.1	Coding geometrical information	191
8.4.2	Coding of functional information	192
8.4.3	Mapping between reference and physical element	193
8.4.4	Construction of local and global systems	198
8.4.5	Boundary conditions prescription	201
8.5	Integration in time	203
8.6	A complete example	206
9	The finite volume method	213
9.1	Some basic principles	214
9.2	Construction of control volumes for vertex-centered schemes	216

9.3	Discretization of a diffusion-transport-reaction problem	219
9.4	Analysis of the finite volume approximation	221
9.5	Implementation of boundary conditions	222
10	Spectral methods	225
10.1	The spectral Galerkin method for elliptic problems	225
10.2	Orthogonal polynomials and Gaussian numerical integration	229
10.2.1	Orthogonal Legendre polynomials	229
10.2.2	Gaussian integration	232
10.2.3	Gauss-Legendre-Lobatto formulae	233
10.3	G-NI methods in one dimension	236
10.3.1	Algebraic interpretation of the G-NI method	237
10.3.2	Conditioning of the stiffness matrix in the G-NI method	239
10.3.3	Equivalence between G-NI and collocation methods	240
10.3.4	G-NI for parabolic equations	243
10.4	Generalization to the two-dimensional case	245
10.4.1	Convergence of the G-NI method	246
10.5	G-NI and SEM-NI methods for a one-dimensional model problem	254
10.5.1	The G-NI method	255
10.5.2	The SEM-NI method	258
10.6	Spectral methods on triangles and tetrahedra	261
10.7	Exercises	265
11	Discontinuous element methods (DG and mortar)	267
11.1	The discontinuous Galerkin method (DG) for the Poisson problem	267
11.1.1	Numerical results for the DG approximation of Poisson problem	272
11.2	The mortar method	273
11.2.1	Characterization of the space of constraints by spectral elements	275
11.2.2	Characterization of the space of constraints by finite elements	276
11.3	Mortar formulation for the Poisson problem	277
11.4	Choosing basis functions	278
11.5	Choosing quadrature formulae for spectral elements	280
11.6	Choosing quadrature formulae for finite elements	281
11.7	Solving the linear system of the mortar method	282
11.8	The mortar method for combined finite and spectral elements	283
11.9	Generalization of the mortar method to multi-domain decompositions	285
11.10	Numerical results for the mortar method	286
12	Diffusion-transport-reaction equations	291
12.1	Weak problem formulation	291
12.2	Analysis of a one-dimensional diffusion-transport problem	294
12.3	Analysis of a one-dimensional diffusion-reaction problem	299

12.4	Finite elements and finite differences (FD)	301
12.5	The mass-lumping technique	302
12.6	Decentred FD schemes and artificial diffusion	304
12.7	Eigenvalues of the diffusion-transport equation	307
12.8	Stabilization methods	309
12.8.1	Artificial diffusion and decentred finite element schemes	310
12.8.2	The Petrov-Galerkin method	312
12.8.3	The artificial diffusion and streamline-diffusion methods in the two-dimensional case	313
12.8.4	Consistency and truncation error for the Galerkin and generalized Galerkin methods	314
12.8.5	Symmetric and skew-symmetric part of an operator	315
12.8.6	Strongly consistent methods (GLS, SUPG)	316
12.8.7	On the choice of the stabilization parameter τ_K	319
12.8.8	Analysis of the GLS method	321
12.8.9	Stabilization through bubble functions	327
12.9	DG methods for diffusion-transport equations	329
12.10	Mortar methods for the diffusion-transport equations	330
12.11	Some numerical tests	332
12.12	An example of goal-oriented adaptivity	334
12.13	Exercises	336
13	Finite differences for hyperbolic equations	339
13.1	A scalar transport problem	339
13.1.1	An a priori estimate	341
13.2	Systems of linear hyperbolic equations	343
13.2.1	The wave equation	345
13.3	The finite difference method	346
13.3.1	Discretization of the scalar equation	347
13.3.2	Discretization of linear hyperbolic systems	349
13.3.3	Boundary treatment	350
13.4	Analysis of the finite difference methods	350
13.4.1	Consistency and convergence	350
13.4.2	Stability	351
13.4.3	Von Neumann analysis and amplification coefficients	356
13.4.4	Dissipation and dispersion	360
13.5	Equivalent equations	364
13.5.1	The upwind scheme case	364
13.5.2	The Lax-Friedrichs and Lax-Wendroff case	367
13.5.3	On the meaning of coefficients in equivalent equations	367
13.5.4	Equivalent equations and error analysis	368
13.6	Exercises	369
14	Finite elements and spectral methods for hyperbolic equations	371
14.1	Temporal discretization	371

14.1.1	The forward and backward Euler schemes	371
14.1.2	The upwind, Lax-Friedrichs and Lax-Wendroff schemes	373
14.2	Taylor-Galerkin schemes	378
14.3	The multi-dimensional case	382
14.3.1	Semi-discretization: strong and weak treatment of the boundary conditions	382
14.3.2	Temporal discretization	385
14.4	Discontinuous finite elements	388
14.4.1	The one-dimensional upwind DG method	388
14.4.2	The multi-dimensional case	393
14.4.3	DG method with jump stabilization	395
14.5	Approximation using spectral methods	396
14.5.1	The G-NI method in a single interval	397
14.5.2	The DG-SEM-NI method	401
14.6	Numerical treatment of boundary conditions for hyperbolic systems	402
14.6.1	Weak treatment of boundary conditions	406
14.7	Exercises	408
15	Nonlinear hyperbolic problems	409
15.1	Scalar equations	409
15.2	Finite difference approximation	414
15.3	Approximation by discontinuous finite elements	415
15.3.1	Temporal discretization of DG methods	418
15.4	Nonlinear hyperbolic systems	424
16	Navier-Stokes equations	429
16.1	Weak formulation of Navier-Stokes equations	431
16.2	Stokes equations and their approximation	435
16.3	Saddle-point problems	439
16.3.1	Problem formulation	439
16.3.2	Analysis of the problem	440
16.3.3	Galerkin approximation, stability and convergence analysis	444
16.4	Algebraic formulation of the Stokes problem	447
16.5	An example of stabilized problem	451
16.6	A numerical example	453
16.7	Time discretization of Navier-Stokes equations	455
16.7.1	Finite difference methods	456
16.7.2	Characteristics (or Lagrangian) methods	458
16.7.3	Fractional step methods	459
16.8	Algebraic factorization methods and preconditioners for saddle-point systems	462
16.9	Free surface flow problems	468
16.9.1	Navier-Stokes equations with variable density and viscosity	469
16.9.2	Boundary conditions	470
16.9.3	Application to free surface flows	471

16.10	Interface evolution modelling	473
16.10.1	Explicit interface descriptions	473
16.10.2	Implicit interface descriptions	473
16.11	Finite volume approximation	478
16.12	Exercises	481
17	Optimal control of partial differential equations	483
17.1	Definition of optimal control problems	483
17.2	A control problem for linear systems	485
17.3	Some examples of optimal control problems for the Laplace equation	486
17.4	On the minimization of linear functionals	487
17.5	The theory of optimal control for elliptic problems	489
17.6	Some examples of optimal control problems	494
17.6.1	A Dirichlet problem with distributed control	494
17.6.2	A Neumann problem with distributed control	495
17.6.3	A Neumann problem with boundary control	495
17.7	Numerical tests	496
17.8	Lagrangian formulation of control problems	502
17.8.1	Constrained optimization in \mathbb{R}^n	502
17.8.2	The solution approach based on the Lagrangian	503
17.9	Iterative solution of the optimal control problem	505
17.10	Numerical examples	510
17.10.1	Heat dissipation by a thermal fin	510
17.10.2	Thermal pollution in a river	512
17.11	A few considerations about observability and controllability	514
17.12	Two alternative paradigms for numerical approximation	516
17.13	A numerical approximation of an optimal control problem for advection–diffusion equations	517
17.13.1	The strategies “optimize–then–discretize” and “discretize–then–optimize”	519
17.13.2	A posteriori error estimates	520
17.13.3	A test problem on control of pollutant emission	523
17.14	Exercises	525
18	Domain decomposition methods	527
18.1	Some classical iterative DD methods	528
18.1.1	Schwarz method	528
18.1.2	Dirichlet-Neumann method	530
18.1.3	Neumann-Neumann algorithm	532
18.1.4	Robin-Robin algorithm	532
18.2	Multi-domain formulation	533
18.2.1	The Steklov-Poincaré operator	533
18.2.2	Equivalence between Dirichlet-Neumann and Richardson methods	535
18.3	Finite element approximation	538

18.3.1	The Schur complement	541
18.3.2	The discrete Steklov-Poincaré operator	543
18.3.3	Equivalence between Dirichlet-Neumann and Richardson methods in the discrete case	545
18.4	Generalization to the case of many subdomains	547
18.4.1	Some numerical results	550
18.5	DD preconditioners in case of many subdomains	551
18.5.1	Jacobi preconditioner	552
18.5.2	Bramble-Pasciak-Schatz preconditioner	554
18.5.3	Neumann-Neumann preconditioner	555
18.5.4	FETI (Finite Element Tearing & Interconnecting) methods	559
18.5.5	FETI-DP (Dual Primal FETI) methods	563
18.5.6	BDDC (Balancing Domain Decomposition with Constraints) methods	566
18.6	Schwarz iterative methods	566
18.6.1	Algebraic form of Schwarz method for finite element discretizations	567
18.6.2	Schwarz preconditioners	569
18.6.3	Two-level Schwarz preconditioners	573
18.7	An abstract convergence result	576
18.8	Interface conditions for other differential problems	577
18.9	Exercises	580
19	Reduced basis approximation for parametrized partial differential equations	585
19.1	Elliptic coercive parametric PDEs	587
19.1.1	Two simple examples	588
19.2	Main components of computational reduction techniques	590
19.3	The reduced basis method	593
19.3.1	RB Spaces	594
19.3.2	Galerkin projection	594
19.3.3	Offline-Online computational procedure	596
19.4	Algebraic and geometric interpretations of the RB problem	597
19.4.1	Algebraic interpretation of the (G-RB) problem	598
19.4.2	Geometric interpretation of the (G-RB) problem	600
19.4.3	Alternative formulations: Least-Squares and Petrov-Galerkin RB problems	602
19.5	Construction of reduced spaces	605
19.5.1	Greedy algorithm	605
19.5.2	Proper Orthogonal Decomposition	608
19.6	Convergence of RB approximations	611
19.6.1	A priori convergence theory: a simple case	611
19.6.2	A priori convergence theory: greedy algorithms	612
19.7	A posteriori error estimation	615
19.7.1	Some preliminary estimates	615

19.7.2	Error bounds	616
19.8	Non-compliant problems	617
19.9	Parametrized geometries and operators	619
19.9.1	Physical parameters	620
19.9.2	Geometrical parameters	622
19.10	A working example: a diffusion-convection problem	626
References		635
Index		649