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Hiroshi Kunita

Stochastic Flows and Jump-Diffusions

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Hiroshi Kunita

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Hiroshi Kunita
Kyushu University (emeritus)
Fukuoka, Japan

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This book is dedicated to my family

Preface

The Wiener process and the Poisson random measure are fundamental to the study of stochastic processes; the former describes a continuous random evolution, and the latter describes a random phenomenon that occurs at a random time. It was shown in the 1940s that any Lévy process (process with independent increments) is represented by a Wiener process and a Poisson random measure, called the Lévy–Itô representation. Further, Itô defined a stochastic differential equation (SDE) based on a Wiener process. He defined also an equation based on a Wiener process and a Poisson random measure. In this monograph, we wish to present a modern treatment of SDE and diffusion and jump-diffusion processes. In the first part, we will show that solutions of SDE will define stochastic flows of diffeomorphisms. Then, we discuss the relation between stochastic flows and heat equations. Finally, we will investigate fundamental solutions of these heat equations (heat kernels), through the study of the Malliavin calculus.

It seems to be traditional that diffusion processes and jump processes are discussed separately. For the study of the diffusion process, theory of partial differential equations is often used, and this fact has attracted a lot of attention. On the other hand, the study of jump processes has not developed rapidly. One reason might be that for the study of jump processes, we could not find effective tools in analysis such as the partial differential equation in diffusion processes. However, recently, the Malliavin calculus for Poisson random measure has been developed, and we can apply it to some interesting problems of jump processes.

A purpose of this monograph is that we present these two theories simultaneously. In each chapter, we start from continuous processes and then proceed to processes with jumps. In the first half of the monograph, we present the theory of diffusion processes and jump-diffusion processes on Euclidean spaces based on SDEs. The basic tools are Itô's stochastic calculus. In Chap. 3, we show that solutions of these SDEs define stochastic flows of diffeomorphisms. Then in Chap. 4, relations between a diffusion (or jump-diffusion) and a heat equation (or a heat equation associated with integro-differential equation) will be studied through properties of stochastic flow.

In the latter half of the monograph, we will study the Malliavin calculus on the Wiener space and that on the space of Poisson random measure. These two types of calculus are quite different in detail, but they have some interesting things in common. These will be discussed in Chap. 5. Then in Chap. 6, we will apply the Malliavin calculus to diffusions and jump-diffusions determined by SDEs. We will obtain smooth densities for transition functions of various types of diffusions and jump-diffusions. Further, we show that these density functions are fundamental solutions for various types of heat equations and backward heat equations; thus, we construct fundamental solutions for heat equations and backward heat equations, independent of the theory of partial differential equations. Finally, SDEs on subdomains of a Euclidean space and those on manifolds will be discussed at the end of Chaps. 6 and 7.

Acknowledgements Most part of this book was written when the author was working on Malliavin calculus for jump processes jointly with Masaaki Tsuchiya and Yasushi Ishikawa in 2014–2017. Discussions with them helped me greatly to make and rectify some complicated norm estimations, which cannot be avoided for getting the smooth density. The contents of Sects. 5.5, 5.6, and 5.7 overlap with the joint work with them [46]. Further, Ishikawa read Chap. 7 carefully and gave me useful advice. I wish to express my thanks to them both. I would also like to thank the anonymous referees and a reviewer, who gave me valuable advices for improving the draft manuscript. Finally, it is my pleasure to thank Masayuki Nakamura, editor at Springer, who helped me greatly toward the publication of this book.

Fukuoka, Japan Hiroshi Kunita

Introduction

Stochastic differential equations (SDEs) based on Wiener processes have been studied extensively, after Itô's work in the 1940s. One purpose was to construct a diffusion process satisfying the Kolmogorov equation. Results may be found in monographs of Stroock–Varadhan [109], Ikeda–Watanabe [41], Oksendal [90], and Karatzas–Shreve [55]. Later, the geometric property of solutions was studied. It was shown that solutions of an SDE based on a Wiener process define a stochastic flow of diffeomorphisms [59].

In 1978, Malliavin [77] introduced an infinite-dimensional differential calculus on a Wiener space. The theory had an interesting application to solutions of SDEs based on the Wiener process. He applied the theory to the regularity of the heat kernel for hypo-elliptic differential operators. Then, the Malliavin calculus developed rapidly. Contributions were made by Bismut [9], Kusuoka–Stroock [69, 71], Ikeda–Watanabe [40, 41], Watanabe [116, 117], and many others.

At the same period, the Malliavin calculus for jump processes was studied in parallel (see Bismut [10], Bichteler–Gravereau–Jacod [7], and Leandre [74]). Later, Picard [92] proposed another approach to the Malliavin calculus for jump processes. Instead of the Wiener space, he developed the theory on the space of Poisson random measure and got a smooth density for the law of a "nondegenerate" jump Markov process. Then, Ishikawa–Kunita [45] combined these two theories and got a smooth density for the law of a nondegenerate jump-diffusion. Thus, the Malliavin calculus can be applied to a large class of SDEs.

In this monograph, we will study two types of SDEs defined on Euclidean space and manifolds. One is a continuous SDE based on a Wiener process and smooth coefficients. We will define the SDE by means of Fisk–Stratonovich symmetric integrals, since its solution has nice geometric properties. The other is an SDE with jumps based on the Wiener process and the Poisson random measure, where coefficients for the continuous part are smooth vector fields and coefficients for jump parts are diffeomorphic maps. These two SDEs are our basic objects of study. We want to show that both of these SDEs generate stochastic flows of diffeomorphisms and these stochastic flows define diffusion processes and jump-diffusion processes. In the course of the argument, we will often consider backward processes, i.e., stochastic processes describing the backward time evolution. It will be shown that inverse maps of a stochastic flow (called the inverse flow) define a backward Markov process. Further, we show that the dual process is a backward Markov process and it can be defined directly by the inverse flow through an exponential transformation. In each chapter, we will start with topics related to Wiener processes and then proceed to those related to Poisson random measures. Investigating these two subjects together, we can understand both the Wiener processes and Poisson random measures more strongly.

Chapters 1 and 2 are preliminaries. In Chap. 1, we propose a method of studying the smooth density of a given distribution, through its characteristic function (Fourier transform); it will be applied to the density problem of infinitely divisible distributions. Then, we introduce some basic stochastic processes and backward stochastic processes. These include Wiener processes, Poisson random measures, martingales, and Markov processes. In Chap. 2, we discuss stochastic integrals. Itô integrals and Fisk–Stratonovich symmetric integrals based on continuous martingales and Wiener processes are defined, and Itô's formulas are presented. Then, we define stochastic integrals based on (compensated) Poisson random measures. Further, we will give L^p-estimates of these stochastic integrals; these estimates will be used in Chaps. 3 and 6 for checking that stochastic flows have some nice properties. The backward stochastic integrals will also be discussed.

In Chap. 3, we will revisit SDEs and stochastic flows, which were discussed by the author [59, 60]. A continuous SDE on d-dimensional Euclidean space $\mathbb{R}^d$ based on a d'-dimensional Wiener process $(W_t^1, \dots, W_t^{d'})$ is given by

$$dX_t = \sum_{k=1}^{d'} V_k(X_t, t) \circ dW_t^k + V_0(X_t, t)\, dt, \tag{1}$$

where $\circ dW_t^k$ denotes the symmetric integral based on the Wiener process W_t^k. If coefficients $V_k(x, t), k = 0, \dots, d$ are of $C_b^{\infty,1}$-class, it is known that the family of solutions $\{X_t^{x,s}, s < t\}$ of the SDE, starting from x at time s, have a modification $\{\Phi_{s,t}(x), s < t\}$, which is continuous in s, t, x and satisfies (a) $\Phi_{s,t} : \mathbb{R}^d \to \mathbb{R}^d$ are C^∞-diffeomorphisms, (b) $\Phi_{s,u} = \Phi_{t,u} \circ \Phi_{s,t}$ for any $s < t < u$ almost surely, and (c) $\Phi_{s,t}$ and $\Phi_{t,u}$ are independent. $\{\Phi_{s,t}\}$ is called a continuous stochastic flow of diffeomorphisms defined by the SDE.

A similar problem was studied for SDE based on the Wiener process and the Poisson random measure. Let $N(dt\, dz)$ be a Poisson random measure on the space $\mathbb{U} = [0, T] \times (\mathbb{R}^{d'} \setminus \{0\})$ with the intensity measure $n(dt\, dz) = dt\nu(dz)$, where ν is a Lévy measure having a "weak drift." We consider an SDE driven by a Wiener process and Poisson random measure:

$$dX_t = \sum_{k=1}^{d'} V_k(X_t, t) \circ dW_t^k + V_0(X_t, t)\, dt + \int_{|z|>0+} (\phi_{t,z}(X_{t-}) - X_{t-}) N(dt\, dz), \tag{2}$$

where $\{\phi_{t,z}\}$ is a family of diffeomorphic maps on $\mathbb{R}^d$ with some regularity conditions (precise conditions will be stated in Sect. 3.2). It was shown that solutions define a stochastic flow of diffeomorphisms stated above (Fujiwara–Kunita [30]). In this monograph, we will prove these facts through discussions of the "master equation" and "backward SDE." By using them, some complicated arguments in previous works [30, 59] are simplified. Further, we will define a backward SDE based on a Wiener process and another based on Wiener process and Poisson random measure. These backward SDEs define backward stochastic flows of diffeomorphisms.

The solution of an SDE (or a backward SDE) based on a Wiener process defines a diffusion process (continuous strong Markov process) (or backward diffusion process). Further, the solution of an SDE (or backward SDE) based on a Wiener processes and a Poisson random measure defines a jump-diffusion (or a backward jump-diffusion). We will study these diffusion and jump-diffusion processes. Let $\{P_{s,t}\}$ be the semigroup defined by $P_{s,t}f(x) = E[f(\Phi_{s,t}(x))]$. In the case of a diffusion process on $\mathbb{R}^d$, its generator is given by a second-order differential operator

$$A(t)f(x) = \frac{1}{2}\sum_{k=1}^{d'} V_k(t)^2 f(x) + V_0(t)f(x), \tag{3}$$

where $V_k(t), k = 0, 1, \ldots, m$ are first-order partial differential operators defined by $V_k(t)f(x) = \sum_i V_k^i(x,t)\frac{\partial}{\partial x_i}f(x)$. In the case of a jump-diffusion process on $\mathbb{R}^d$, the generator is given by an integro-differential operator of the form

$$A_J(t)f = \frac{1}{2}\sum_{k=1}^{d'} V_k(t)^2 f(x) + V_0(t)f(x) + \int_{|z|>0+} \{f(\phi_{t,z}(x)) - f(x)\}\nu(dz), \tag{4}$$

where the last integral is an improper integral.

In Chap. 4, we study the relation between stochastic flows and time-dependent heat equations and backward heat equations associated with the differential operator $A(t)$ of (3) and integro-differential operator $A_J(t)$ of (4), respectively. For a given time t_1 and a bounded *smooth* function $f_1(x)$, the function $v(x,s) := P_{s,t_1}f_1(x) = E[f_1(\Phi_{s,t_1}(x))]$ is a smooth function of x. Further, in the case of diffusions, $v(x,s)$ is the unique solution of the final value problem of the time-dependent backward heat equation:

$$\frac{\partial}{\partial s}v(x,s) = -A(s)v(x,s) \ \text{ for } s < t_1, \quad v(t_1,x) = f_1(x). \tag{5}$$

This fact will be extended to a more general class of the operator $A(t)$. Consider

$$A^{\mathbf{c}}(t)f = \frac{1}{2}\sum_{k=1}^{d'}(V_k(t) + c_k(t))^2 f + (V_0 + c_0)f, \tag{6}$$

where $\mathbf{c} = (c_k(x,t); k = 0, 1, \ldots, m)$ are bounded smooth functions. We show that the semigroup with the generator $A^{\mathbf{c}}(t)$ is obtained by an exponential transformation based on $c_k(x,t)$; it is given by $P^{\mathbf{c}}_{s,t} f(x) = E[f(\Phi_{s,t}(x)) G_{s,t}(x)]$, where

$$G_{s,t}(x) = \exp\Big\{ \sum_{k \geq 1} \int_s^t c_k(\Phi_{s,r}(x), r) \circ dW_r^k + \int_s^t c_0(\Phi_{s,r}(x)\, dr \Big\}.$$

Further, $v(x,s) := P^{\mathbf{c}}_{s,t_1} f_1(x)$ is the unique solution of the final value problem (5) associated with the operator $A^{\mathbf{c}}(t)$. For jump-diffusion processes, we will also extend the integro-differential operator $A_J(t)$ to another one, which will be denoted by $A_J^{\mathbf{c},\mathbf{d}}(t)$. Then, we will study the final value problem of the backward heat equation associated with the operator $A_J^{\mathbf{c},\mathbf{d}}(t)$ (see Sect. 4.5).

We are also interested in the initial value problem of the time-dependent heat equation associated with the operator $A^{\mathbf{c}}(t)$ given by (6):

$$\frac{\partial}{\partial t} u(t,x) = A^{\mathbf{c}}(t) u(t,x) \ \text{ for } t > t_0, \quad u(t_0, x) = f_0(x). \tag{7}$$

For this problem, we solve SDE (1) to the backward direction and obtain a backward stochastic flow $\{\check{\Phi}_{s,t}\}$; then, we define a backward semigroup by $\check{P}^{\mathbf{c}}_{s,t} f(x) := E\big[f(\check{\Phi}_{s,t}(x)) \check{G}_{s,t}(x)\big]$, where $\check{G}_{s,t}(x)$ is the exponential functional associated with the backward flow $\{\check{\Phi}_{s,t}\}$. Then, if $f_0(x)$ is a bounded *smooth* function, the solution of the forward equation (7) exists uniquely, and it is represented by $u(x,t) = \check{P}^{\mathbf{c}}_{t_0,t} f_0(x)$.

We stress that the final function f_1 (or initial function f_0) is smooth in these studies. Indeed, the smoothness of functions $v(x,s) = P^{\mathbf{c}}_{s,t} f_1(x)$, etc. with respect to x follows from the smoothness of $f_1(x)$ and the stochastic flow $\Phi_{s,t}(x)$ with respect to x, a.s. If the function f_1 is not smooth, we need additional arguments for the solution of equations (5) and (7), which will be discussed in Chap. 6 using the Malliavin calculus.

Another subject of Chap. 4 is the investigation of the dual of a given diffusion and a jump-diffusion with respect to the Lebesgue measure. It will be shown that the dual of these processes can be constructed through the change-of-variable formula concerning stochastic flows $\{\Phi_{s,t}\}$; the stochastic process defined by the inverse maps $\check{X}_s = \check{\Psi}_{s,t}(x) = \Phi_{s,t}^{-1}(x)$ should be a dual process of $X_t = \Phi_{s,t}(x)$, and it is a backward diffusion or a backward jump-diffusion with respect to s, where t is the initial time of the process. The dual semigroup of the semigroup $\{P_{s,t}\}$ is then defined by using the inverse flow $\{\check{\Psi}_{s,t}\}$ as

$$P^*_{s,t} g(x) = E[g(\check{\Psi}_{s,t}(x)) \det \nabla \check{\Psi}_{s,t}(x)],$$

where $\nabla \check{\Psi}_{s,t}$ is the Jacobian matrix of the diffeomorphism $\check{\Psi}_{s,t}; \mathbb{R}^d \to \mathbb{R}^d$.

In the latter half of this monograph, we will study the Malliavin calculus on the Wiener space and the space of Poisson random measure, called Poisson space; we will apply it for proving the existence of fundamental solutions for heat equations discussed in Chap. 4.

In Chap. 5, we will discuss the Malliavin calculus on the Wiener space and that on the Poisson space (space of Poisson random measure) separately. Then, we will combine these two. For the Malliavin calculus on the Wiener space, we will restrict our attention to the problem of finding the smooth density for laws of Wiener functionals. Our discussion is motivated by Malliavin and Watanabe (Watanabe [116, 117]), but we will take a simple and direct approach; we will not consider the Ornstein–Uhlenbeck semigroup on the Wiener space. Instead, we study the derivative operator D_t and its adjoint δ (Skorohod integral) directly. Then, we give an estimate of Skorohod integrals using L^p-Sobolev norms; a new proof is given for Theorem 5.2.1. A criterion for the smooth density of laws of Wiener functional will be given in terms of the celebrated "Malliavin covariance" in Sect. 5.3.

Another reason why we do not follow Ornstein–Uhlenbeck semigroup argument is that a similar fact is not known for Poisson space; in fact, we want to bring together the Malliavin calculus on the Wiener space and that on the Poisson space in a unified method. In Sects. 5.4, 5.5, 5.6, 5.7, and 5.8, we will discuss the Malliavin calculus on the Poisson space, which is characterized by a Lévy measure ν. A basic assumption for the Poisson random measure is that the characterizing Lévy measure is nondegenerate and satisfies the order condition at the center. Here, the origin 0 is regarded as the center of the Lévy measure defined on $\mathbb{R}^{d'} \setminus \{0\}$. We will see that the difference operator $\tilde{D}_u$ and its adjoint operator $\tilde{\delta}$ defined by Picard [92] work well as D_t and δ do on the Wiener space.

Criteria of the smooth density for Poisson functionals are more complicated. We need a family of L^p-Sobolev norms conditioned to a family of neighborhoods of the center of the Lévy measure. It will be given in Sects. 5.5, 5.6, 5.7, and 5.8. Further, in Sects. 5.9, 5.10, and 5.11, we will study the Malliavin calculus on the product of the Wiener space and the Poisson space. A criterion for the smooth density of the law of a Wiener–Poisson functional will be given after introducing the "Malliavin covariance at the center."

In the application of the Malliavin calculus to solutions of an SDE, properties of stochastic flows defined by the SDE are needed. In Chap. 6, we study the existence of smooth densities of laws of a nondegenerate diffusion and a nondegenerate jump-diffusion defined on a Euclidean space. The class of nondegenerate diffusions includes elliptic diffusions and hypo-elliptic diffusions. Further, the class of non-degenerate jump-diffusions includes pseudo-elliptic jump-diffusions. Let $P^{\mathbf{c}}_{s,t}$ (or $P^{\mathbf{c},\mathbf{d}}_{s,t}$) be the semigroup associated with the generator $A^{\mathbf{c}}(t)$ (or $A^{\mathbf{c},\mathbf{d}}_J(t)$). It will be shown that its transition functions $P^{\mathbf{c}}_{s,t}(x,\cdot)$ (or $P^{\mathbf{c},\mathbf{d}}_{s,t}(x,\cdot)$) have densities $p^{\mathbf{c}}_{s,t}(x,y)$ (or $p^{\mathbf{c},\mathbf{d}}_{s,t}(x,y)$), which are smooth with respect to both variables x and y, and further, the family of the densities is the fundamental solution of the backward heat equation associated with the operator $A^{\mathbf{c}}(t)$ (or $A^{\mathbf{c},\mathbf{d}}_J(t)$); the fundamental solution of the heat equation will be obtained as a family of density functions of a backward transition function $\check{P}^{\mathbf{c}}_{s,t}(x, E)$ associated with the semigroup $\{\check{P}^{\mathbf{c}}_{s,t}\}$, etc. Thus, initial–final value problems (5) and (7) are solved for any bounded continuous functions f_0 and f_1, respectively.

In Sects. 6.7 and 6.8, for elliptic diffusions and pseudo-elliptic jump-diffusions, we will discuss the short-time asymptotics of the transition density functions as $t \downarrow s$, making use of the Malliavin calculus. Our Malliavin calculus cannot be applied to jump processes or jump-diffusion processes which admit big jumps. Indeed, in order to apply the Malliavin calculus, solution X_t of the SDE should be at least an element of $L^{\infty-} = \bigcap_{p>1} L^p$, and the fact excludes solutions of SDEs with big jumps. In Sect. 6.9, we consider such processes: we first truncate big jumps and get the smooth density. Then, we add big jumps and show that this action should preserve the smooth density, where the short-time asymptotics of the fundamental solution will be utilized.

It is hard to apply the Malliavin calculus directly to (jump) diffusions on a bounded domain of a Euclidean space or those defined on a manifold. In Sect. 6.10, we consider a process killed outside of a bounded domain of a Euclidean space. In order to get a smooth density for the killed process, we need two facts. One is a short-time estimate of the density of a non-killed process. The other is a potential theoretic argument of a strong Markov process using hitting times. We show that the density function $q^{\mathbf{c}}_{s,t}(x, y)$ of the killed transition function is smooth with respect to x and y; further, we show that $q^{\mathbf{c}}_{s,t}(x, y)$ is the fundamental solution for the backward heat equation (5) on an arbitrary bounded domain of a Euclidean space with the Dirichlet boundary condition.

Finally, in Chap. 7, we study SDEs on a manifold. Stochastic flow generated by an SDE on a manifold will be discussed in Sect. 7.1. Diffusions, jump-diffusions, and their duals will be treated in Sects. 7.2 and 7.3. Then, the smooth density for a (jump) diffusion on a manifold will be obtained by piecing together killed densities on local charts. It will be discussed in Sects. 7.4 and 7.5.

A Guide for Readers Discussions of the Malliavin calculus for Poisson random measures contain some complicated and technical arguments. For the beginner or the reader who is mainly interested in Wiener processes and diffusion processes, we suggest to skip these arguments at the first reading. After Chap. 4, we could proceed in the following way:

Chapter 5, Sects. 5.1, 5.2, 5.3 $\longrightarrow$ Chap. 6, Sects. 6.1, 6.2, 6.3, 6.7, 6.8, 6.10 $\longrightarrow$ Chap. 7, Sects. 7.1, 7.2, 7.3, 7.4.

The author hopes that this course should be readable.

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