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Alin Bostan
Kilian Raschel *Editors*

Transcendence in Algebra, Combinatorics, Geometry and Number Theory

TRANS19 – Transient Transcendence in Transylvania, Brașov, Romania, May 13–17, 2019
Revised and Extended Contributions

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Forewords

The general topic of this volume is the emergence of transcendence in various fields of mathematics, such as algebra, combinatorics, geometry and number theory. The volume is composed of 23 chapters, which we grouped into five main thematic parts:

(i) Transcendence in algebraic geometry
(ii) Transcendence in combinatorics and physics
(iii) Transcendence in commutative algebra
(iv) Transcendence in computer algebra
(v) Transcendence in number theory

This is in the natural continuation of the conference Transcendence in Transylvania (aka Trans'19) that we co-organized in May 2019 in Braşov, Romania. The conference gathered international experts from various fields of mathematics and computer science, with diverse interests and viewpoints on transcendence.

A Detour via Gessel's Lattice Path Conjecture. Being (in part) experimental mathematicians, we wish to present our own experience with transcendence on one example, namely Gessel's lattice path conjecture. It reads as follows: Gessel walks are lattice paths confined to the quarter plane $\mathbb{N}^2$, that start at the origin (0, 0) and move by unit steps in one of the following directions: West, East, South-West and North-East; see Fig. 1. Gessel excursions are those Gessel walks which return to the origin. They have been puzzling the combinatorics community since 2001, when Ira Gessel conjectured that for all $n \geq 0$, the excursions (named after him) of even length $2n$ are counted by the hypergeometric formula

$$e(2n) = 16^n \frac{(5/6)_n (1/2)_n}{(2)_n (5/3)_n}, \tag{1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ denotes the Pochhammer symbol. (Obviously, there are no Gessel excursions of odd length, that is $e(2n+1) = 0$ for

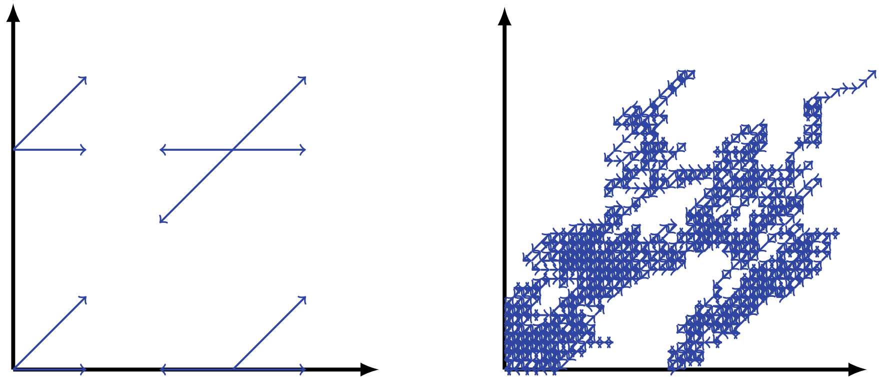

Fig. 1. Gessel walks: the allowed steps (left) and a random Gessel walk (right)

all $n \geq 0$.) In 2008, Kauers, Koutschan and Zeilberger provided a computer-aided proof of this conjecture [KKZ09], which involved computer algebra tools and massive calculations. Apart from its striking simplicity, the sequence $(e(2n))_{n\geq 0}$ in (1) has a nice property: its generating function $\sum_{n\geq 0} e(2n)t^n$ is algebraic. Both facts urge for (direct) combinatorial explanations, which however are still lacking as of 2021.

Now, more generally, for $(i,j) \in \mathbb{N}^2$ and $n \geq 0$, let us call $q(i,j;n)$ the number of Gessel walks of length n ending at the point (i,j); in particular, $e(n) = q(0,0;n)$. A further natural question is the following: Is the complete generating function of Gessel walks

$$Q(x,y;t) = \sum_{i,j,n\geq 0} q(i,j;n)x^i y^j t^n \tag{2}$$

transcendental or algebraic? In particular, what is the nature of the generating function $Q(1,1;t) = \sum\limits_{n\geq 0} q(n)t^n$ for the total number *q(n)* of Gessel walks with prescribed length n?

Although these combinatorial statements are just examples of transcendence questions, we feel that they played a special role in the elaboration of the present volume, at least for two reasons. First, Gessel's conjecture has received a lot of attention over the past 20 years and has given rise to many interesting methods and results [KKZ09, BK10, KR11, BKR17, BM16, Bud20]. In fact, it is one instance of a more general question consisting in counting lattice walks confined to a given cone. This is a natural and versatile problem, rich in many applications in algebraic combinatorics, queuing theory [CB83], probability theory [FIM17, DW15] and, of course, in enumerative combinatorics [BMM10] via encodings of numerous discrete objects (e.g. permutations, maps, etc.) by lattice walks. Second, from a more personal point of view, both of us (Alin Bostan and Kilian Raschel, editors of the present volume) have worked on these results, initially independently and using totally different ideas, and eventually joined our efforts.

To make a long story short, after several years of unsuccessful attempts, the complete generating function $Q(x,y;t)$ in (2) was proved to be algebraic by the first editor of this volume (AB) together with Manuel Kauers, using computer algebra techniques [BK10]. In many subsequent works, this result was qualified as a true computational *tour de force* and was listed as one of the achievements of modern computer algebra algorithms [Kal21]. At the same time, the second editor (KR) developed techniques coming from complex analysis and probability theory to compute generating functions such as $Q(x,y;t)$, in terms of (intrinsically transcendent) elliptic functions [KR11]. A few years later, both of us, together with Irina Kurkova, were able to offer the first "human proof" of Gessel's lattice path conjecture [BKR17], proving both the closed-form formula (1) and the algebraicity of the series (2). Interestingly, in this proof, the algebraic function Q is expressed as a finite sum of transcendental quantities (related to Weierstrass zeta functions), but the *transcendence is transient*: each of the transcendental summands decomposes as sums/differences of "smaller" algebraic and transcendental parts, and after telescoping summation the transcendental parts magically vanish.

Transcendence at the Crossroads of Many Mathematical Domains. Beyond Gessel's conjecture, transcendence questions have been asked for many other lattice walk models, leading the mathematical community to systematically study small step walk models; see the pioneering work [BMM10]. Tools from diverse horizons have been used: combinatorics [BMM10], functional equations [BM16], complex analysis [FIM17, KR11, BKR17], probability theory [DW15], computer algebra [KKZ09, BK10], Galois theory of difference equations [DHRS18, DHRS20], number theory [BRS14], etc.

This context strongly encouraged us to organize an international conference bringing together experts from these different communities to answer long-standing conjectures, to learn each other's techniques and to plan directions and investigations for the future.

All talks given at the conference, as well as all 23 contributions to this volume, are related to this concept of transcendence, in close relation with other mathematical areas as described at the beginning of our introduction.

Alin Bostan
Kilian Raschel

Acknowledgments

All papers in this volume were refereed very rigorously. We are deeply grateful to all our referees for their time-consuming effort and discipline in evaluating the articles. We thank Ambrose Berkumans, Elizabeth Loew, Hemanth MVN and Robinson dos Santos from Springer for their enthusiasm and for their valuable assistance. We also address our gratitude to the local organizers Nicuşor Minculete, Eugen Pălănea and Diana Savin, as well as to Faculty of Mathematics and Computer Science of Braşov for their support in the organization of the Trans'19 conference. Finally, we thank Herwig Hauser, Bruno Salvy and Sergey Yurkevich for their careful reading of this foreword and for their useful remarks.

Alin Bostan
Kilian Raschel

Lists and Summaries of Papers

We list the articles collected in this volume, and for each of them, we give a brief summary.

First Part: Transcendence in Algebraic Geometry

In Chapter 1, Masanori Asakura describes an algorithm for computing values of Dwork's p-adic hypergeometric function $\mathcal{F}^{\mathrm{Dw}}_{a,b}(t)$ modulo p^n. The algorithm is based on an explicit description of the Frobenius action on the rigid cohomology of the hypergeometric curve $(1-x^N)(1-y^M)=t$, where M and N are the denominators of the rational numbers a and b. The bit complexity of the new algorithm is $O(n^4(\log n)^3)$. A direct application of Dwork's congruences [Dwork69] leads to exponential complexity in n; hence, the new method compares very favourably.

Given a multivariate Laurent polynomial g whose Newton polytope has one interior point, the generating function of the constant term of the powers of g is known to satisfy Dwork's congruences modulo powers of a prime. In Chapter 2, Frits Beukers provides an example of a matrix version of Dwork's congruences in the case when there are more interior points. The proof relies on a geometric approach to these congruences, appearing in two recent papers [BV20, BV21a] written jointly with Masha Vlasenko. While the hypergeometric functions appearing in Dwork's initial congruences are transcendental, the ones occurring in the new matrix version are algebraic.

In Chapter 3, Thomas Dreyfus, Charlotte Hardouin, Julien Roques and Michael F. Singer investigate the main geometric properties (irreducibility, singularities, genus, uniformization) of the so-called *kernel curve*, an object naturally attached to functional equations satisfied by generating functions of walks in the quarter plane. The classification of these generating functions according to their algebraic and differential properties is an active area of research in combinatorics, to which the same authors contributed with two recent important papers [DHRS18, DHRS20].

In Chapter 4, Gwladys Fernandes surveys hypertranscendence results for the solutions of several famous functional equations: Schröder's equation $f(s\,z) = R(f(z))$, Böttcher's equation $f(z^d) = R(f(z))$ and Abel's equation $f(R(z)) = f(z) + 1$, where R is a rational function. Becker and Bergweiler [BB95] listed all the differentially algebraic solutions of these classes of equations. The proof of this classification result combines various mathematical tools, going from the theory of iteration in complex analysis and dynamical systems, to the algebro-differential notion of coherent families developed by Boshernitzan and Rubel [BR86].

In Chapter 5, Masha Vlasenko gives a short and arithmetically motivated introduction to the definition of the limiting mixed Hodge structures of Deligne and Schmid. In 2016, Golyshev and Zagier [GZ16] proved the first gamma conjecture in mirror symmetry for Fano threefolds of Picard rank 1. Their proof involved computing the so-called *Apéry constants*, related to Apéry's famous proof of irrationality of $\zeta(3)$, for differential operators with maximally unipotent local monodromy. In a recent paper [BV21b], Masha Vlasenko, together with Spencer Bloch, showed that Apéry constants for Picard-Fuchs differential operators are always (explicit) periods, and they linked them to periods of a limiting mixed Hodge structure. The current chapter explains one of the key ingredients used in the proof.

Second Part: Transcendence in Combinatorics and Physics

Chapter 6: *Beck-type identities for Euler pairs of order r*, by Cristina Ballantine and Amanda Welch

Chapter 7: *Quarter-plane lattice paths with interacting boundaries: the Kreweras and reverse Kreweras models*, by Nicholas R. Beaton, Aleksander L. Owczarek and Ruijie Xu

Chapter 8: *Infinite product formulae for generating functions for sequences of squares*, by Christian Krattenthaler, Mircea Merca and Cristian-Silviu Radu

Chapter 9: *A theta identity of Gauss connecting functions from additive and multiplicative number theory*, by Mircea Merca

Chapter 10: *Combinatorial quantum field theory and the Jacobian conjecture*, by Adrian Tanasa

The modern theory of integer partitions has its origins in Euler's discovery in 1748 that the number of partitions of any $n \in \mathbb{N}$ into distinct parts equals the number of partitions of n into odd parts. This is the prototype of all subsequent partition identities, and an elegant proof relies on the identity between transcendental generating functions, $\prod_{n=1}^{\infty}(1+q^n) = \prod_{n=1}^{\infty}\frac{1}{1-q^{2n-1}}$. In 1883, Glaisher found a purely bijective proof of Euler's result [Gla83]. In Chapter 6, Cristina Ballantine and Amanda Welch consider more general partition identities of this type, related to results by Subbarao [Sub71] and Andrews [And17]. They provide both analytic and combinatorial proofs.

In Chapter 7, Nicholas R. Beaton, Aleksander L. Owczarek and Ruijie Xu study lattice walks in the quarter plane with weights associated with visits to the two axes and the origin. They consider two specific models: Kreweras walks (with allowed steps $\{\mathrm{NE}, \mathrm{S}, \mathrm{W}\}$) and reverse Kreweras walks (with allowed steps $\{\mathrm{SW}, \mathrm{N}, \mathrm{E}\}$). Using the so-called *algebraic kernel method*, they prove that for the Kreweras model, the generating function is always D-finite (i.e. solution of a linear differential equation with polynomial coefficients) but possibly transcendental, and for the reverse Kreweras model, the generating function is even algebraic.

Euler's pentagonal number theorem [Eul80] was one of Euler's deepest discoveries. It states the identity between two transcendental power series,

$$\prod_{n\geq 1}(1-q^n) = \sum_{n\geq 0}(-1)^{\lfloor (n+1)/2\rfloor} q^{a_n},$$

where $(a_n)_{n\geq 0} = (0, 1, 2, 5, 7, 12, 15, \ldots)$ is the sequence of integers $m \in \mathbb{N}$ such that $24m+1$ is a square. In Chapter 8, Christian Krattenthaler, Mircea Merca and Cristian-Silviu Radu prove similar product formulas for other transcendental generating functions of the form $\sum_{n\geq 0} \pm q^{a_n}$, for many sequences $(a_n)_{n\geq 0}$ defined by the property that $Pa_n + b^2$ is a perfect square, where P and b are given integers. The proofs rely on the theory of modular functions.

A beautiful identity due to Gauss [Gau66, p. 447, eq. (14)] states that

$$\sum_{n=0}^{\infty} q^{n(n+1)/2} = \prod_{m=1}^{\infty}\frac{1-q^{2m}}{1-q^{2m-1}}.$$

This implies the recurrence relation $\sum_{j=0}^{\infty}(-1)^{j(j+1)/2}\mathrm{pod}(n - j(j+1)/2) = 0$ for all $n > 0$, where $\mathrm{pod}(n)$ is the number of partitions of n into parts not congruent to 2 modulo 4. In Chapter 9, Mircea Merca considers various refinements and generalizations of this identity. They are expressed in terms of the total number $S(k, n)$ of k's in all the partitions of n into parts not congruent to *2* modulo *4*. The proof relies on a truncated theta series identity by Andrews and Merca [AM18].

The Jacobian conjecture states that if a polynomial mapping F from $\mathbb{C}^n$ to itself has Jacobian determinant which is a nonzero constant, then F has a polynomial inverse. In 1980, Wang proved the conjecture when $\deg(F) \leq 2$ [Wan80] and in

1982, Bass, Connell and Wright reduced the general statement to the case $\deg(F) = 3$ [BCW82]. In Chapter 10, Adrian Tanasa gives a short introductory article on the combinatorial quantum field theory (QFT) approach to this notorious open problem, which attempts to fill the gap between the cases of degree 2 and 3. The main result is about reducing the degree of the map involved, but with the caveat that new parameters are introduced and now one has a family of maps to deal with, instead of a single map. This contribution is a summary of the paper [dGST16] by Tanasa with two other collaborators, de Goursac and Sportiello.

Third Part: Transcendence in Commutative Algebra

Chapter 11: *How regular are regular singularities?*, by Herwig Hauser
Chapter 12: *Néron desingularization of extensions of valuation rings*, by Dorin Popescu, with an Appendix by Kęstutis Česnavičius
Chapter 13: *Diagonal representation of algebraic power series: a glimpse behind the scenes*, by Sergey Yurkevich

In Chapter 11, Herwig Hauser offers a conceptual, functional-analytic perspective on regular singular points of linear differential equations with meromorphic coefficients. The general idea is to view an arbitrary regular singular equation as a perturbation of an Euler equation and to construct an isomorphism between their solutions. As the explicit solutions of the Euler equation are well known, one obtains in this way a precise description of the solutions of the original equation. The article sketches how one can recover, using this viewpoint, some classical theorems of Fuchs and Frobenius on the structure of local solutions.

In Chapter 12, Dorin Popescu returns to his celebrated 1986 result [Pop86] on general Néron desingularization for local extensions of valuation rings. In modern language, it says that every regular homomorphism of Noetherian rings is *ind-smooth*. Ind-smoothness is a very useful concept, as it allows, in many concrete problems, to replace a general algebra by a smooth algebra of finite type. The main result of this chapter provides necessary and sufficient conditions for an injective local homomorphism of valuation rings of characteristic zero to be ind-smooth.

In Chapter 13, Sergey Yurkevich gives a detailed account and proof of a celebrated result by Denef and Lipshitz [DL87] that any algebraic power series in n variables can be written as a diagonal of a rational power series in $n + 1$ variables. The proof relies on two important facts: (i) the ring of algebraic power series in n variables is the Henselization of the localization of the ring of polynomials at the maximal ideal generated by the variables; (ii) the Henselization can be described as the direct limit of étale extensions. The article includes a comprehensive proof of these results.

Fourth Part: Transcendence in Computer Algebra

In 1987, David and Gregory Chudnovsky discovered an incredible formula for the number π [CC88]:

$$\pi = \frac{53360\sqrt{640320}}{13591409} \times {}_4\mathrm{F}_3\left(\left[\frac{1}{6},\frac{1}{2},\frac{5}{6},\frac{558731543}{545140134}\right];\left[\frac{13591409}{545140134},1,1\right];-\frac{1}{53360^3}\right)^{-1}.$$

This formula is not only spectacular in its appearance, but also the basis of the fastest algorithm used in practice for computing digits of π. In Chapter 14, Jesús Guillera describes a new method, based on elliptic modular functions, which allows to prove this identity automatically, using computer algebra, in a bunch of seconds on a laptop.

The partition sequence $(p(n))_{n\geq 1} = (1,2,3,5,7,\mathbf{11},15,22,30,42,56,\mathbf{77},\ldots)$ counts the number of distinct ways of representing n as a sum of positive integers. A famous result of Ramanujan [Ram00] asserts that the integer $p(11n+6)$ is divisible by 11 for any $n\geq 0$. In Chapter 15, Mark van Hoeij and Cristian-Silviu Radu describe an algorithm that computes a special basis for the space of modular functions for $\Gamma_0(N) := \left\{\begin{pmatrix} a & b \\ c & d\end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N|c\right\}$ with poles only at infinity. As an application, they obtain a computerized proof of Ramanujan's divisibility result.

A classical result in the theory of modular functions [Sti84, Zag08] asserts that if *h(z)* is a modular function and *g(z)* is a modular form of positive weight, both for a fixed congruence subgroup, and if locally $g(z) = y(h(z))$, then the function *y(t)* is D-finite in *t*. In Chapter 16, Peter Paule and Cristian-Silviu Radu design a guess-and-prove algorithm for finding a linear differential equation satisfied by *y(t)*. This allows for automatic proofs of identities such as $\theta_3(z)^2 = {}_2F_1([1/2,1/2];[1];\lambda(z))$, where $\theta_3(z)$ is Jacobi's theta function $\theta_3(z) = \sum_{n\in\mathbb{Z}} q^{n^2/2} = 1+2\sqrt{q}+2q^2+2q^{9/2}+\cdots \quad (q=\exp(2\pi i z), \mathrm{Im}(z)>0, |q|<1)$ and $\lambda(z)$ is Legendre's elliptic modular function

$$\lambda(z) = \left(\frac{\sum_{n \in \mathbb{Z} + \frac{1}{2}} q^{n^2/2}}{\theta_3(z)} \right)^4 = 16\sqrt{q} - 128q + 704q^{3/2} - 3072q^2 + \cdots.$$

In Chapter 17, Carsten Schneider and Wadim Zudilin illustrate the general algorithmic process of "creative telescoping" in a Diophantine context. As π is transcendental [Lin82], so is $\zeta(4) = \pi^4/90$; however, quantitative versions of this property remain open, even the quantitative version of the irrationality of $\zeta(4)$. The common method for proving upper bounds on the irrationality measure of zeta values is to construct linear forms in 1 and those values, with rational coefficients. The authors give two such constructions of linear forms in 1 and $\zeta(4)$, and they provide a computer-aided proof that the corresponding linear forms are equal.

Fifth Part: Transcendence in Number Theory

The classical Skolem–Mahler–Lech theorem [Lec53] characterizes the possible supports (i.e. sets of indices with nonzero coefficients) of rational power series over a field of characteristic zero: they are the subsets of $\mathbb{N}$ whose characteristic function is eventually periodic. For algebraic power series in characteristic zero, the analogue of the Skolem–Mahler–Lech theorem is still an open problem. In positive characteristic $p > 0$, the situation is much better understood: the support of an algebraic power series is a p-automatic set [Der07]. It can be an arithmetic sequence, e.g. $A(x) := 1/(1-x) = 1 + x + x^2 + \cdots$, but also a geometric sequence, e.g. $G_p(x) := x + x^p + x^{p^2} + \cdots$. In Chapter 18, Jason Bell addresses the characterization of all cases when the support set is a sequence satisfying a linear recurrence with constant coefficients. He proves that they can be built up from $A(x)$ and $G_p(x)$.

A natural question in the field of Diophantine equations is "intersecting linear sequences", which asks to decide whether two linearly recurrent sequences with constant coefficients share only finitely many common values. Given an integer $d > 1$ which is not a square, it is classical that the *Pell equation* $X^2 - dY^2 = \pm 1$

admits infinitely many solutions $(X_n, Y_n) \in \mathbb{N} \times \mathbb{N}$, where both coordinate sequences $(X_n)_n$ and $(Y_n)_n$ satisfy linear recurrences of order 2 with constant coefficients. In Chapter 19, Florian Luca surveys recent results on special cases of the intersection question; they concern the occurrence of interesting arithmetic sequences, such as Fibonacci numbers $(F_n)_{n \geq 0}$, in X-coordinates of Pell equations. A typical example is the following result by the author and Alain Togbé [LT18] *the equation* $X_n = F_m$ *has at most one positive integer solution* (n, m), *except when* $d = 2$, *for which* $X_1 = F_1 = F_2 = 1$ *and* $X_2 = F_4 = 3$.

In algebraic number theory, the *regulator* of an algebraic number field $\mathbb{K}$ is a positive number which measures the "density" of the units in the ring of algebraic integers of $\mathbb{K}$: if the regulator is small, this means that "there are many units". For an imaginary quadratic field $\mathbb{Q}(\sqrt{D})$ with $D<0$, the regulator is 1. For a real quadratic field $\mathbb{Q}(\sqrt{D})$ with $D > 0$, it is the natural logarithm of the fundamental unit of $\mathbb{K}$; for instance, if $d = 1 \mod 4$, it is $\log((a + b\sqrt{d})/2)$, where (a, b) is the smallest solution of the Pell equation $x^2 - dy^2 = \pm 4$. In arbitrary number fields, the regulator is the determinant of some submatrix of the matrix built by the logarithms of absolute values of all conjugates of a fundamental set of units. Units can be completed p-adically, and – at least for CM fields (these are quadratic imaginary extensions of a field which is real in all of its embeddings, also called "totally real") – Leopoldt's p-adic regulator of $\mathbb{K}$ is defined like above, with respect to the p-adic logarithm. The question is: Will this regulator always be non-trivial? Leopoldt's conjecture [Leo62] is a famous open problem, stating that this is the case: the p-adic regulator of a number field does not vanish. The conjecture was proved in 1967 by Brumer [Bru67], using results of Ax and Baker, for abelian number fields (in particular, for quadratic number fields). Since then, the next step was considered to be a proof of the conjecture for CM fields, but not even the case of solvable extensions is known to hold to this day. In Chapter 20, Preda Mihăilescu provides a (conditional) proof of the statement that for odd primes p, Leopold's conjecture holds for CM fields over $\mathbb{Q}$, and herewith he connects the Leopoldt conjecture to another important conjecture of Iwasawa—building a highly unexpected bridge, which he is prepared to use in subsequent work.

E-functions are D-finite and entire power series subject to some arithmetic conditions; they generalize the exponential function. The class contains most D-finite exponential generating functions (EGFs) in combinatorics and many special functions such as the confluent hypergeometric function ${}_1F_1(\alpha; \beta; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n}{(\beta)_n} \frac{z^n}{n!}$, with $\alpha \in \mathbb{Q}$ and $\beta \in \mathbb{Q} \backslash \mathbb{Z}_{<0}$. A non-hypergeometric example is the EGF of the Delannoy numbers

$$D(z) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} \right) \frac{z^n}{n!},$$

which satisfies $zD''(z) + (1 - 6z)D'(z) + (z - 3)D(z) = 0$. In 1949, Siegel asked [Sie49, Ch. II, Sect. 9] whether any E-function can be expressed in terms of (generalized) confluent hypergeometric series. For instance, $D(z)$ is equal to

$e^{(3-2\sqrt{2})z} \cdot {}_1F_1(1/2\,;1;4\sqrt{2}z)$. In 2004, Gorelov proved that Siegel's question admits a positive answer when the given E-function $f(z)$ satisfies a linear differential equation of order 2 [Gor04]. In Chapter 21, Julien Roques and Tanguy Rivoal provide a new proof of this result. More precisely, they show that $f(z)$ can be written $f(z) = a(z)e^{\mu z}{}_1F_1(\alpha;\beta;\lambda z) + b(z)e^{\mu z}{}_1F_1'(\alpha;\beta;\lambda z)$, where $a(z), b(z) \in \bar{\mathbb{Q}}(z)$, $\lambda \in \bar{\mathbb{Q}}^{\times}$, $\mu \in \bar{\mathbb{Q}}$, and $\alpha \in \mathbb{Q}$, $\beta \in \mathbb{Q}\backslash\mathbb{Z}_{\leq 0}$ are such that $\alpha - \beta \notin \mathbb{Z}$.

In Chapter 22, Jonathan Sondow[1] studies the irrationality and the transcendence of alternating series of two types (I and II); these were introduced by Euler [Eul88, Chap. XVIII], who gave recipes for converting them into equivalent continued fractions. For instance, Euler showed that Leibniz's alternating series $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$ can be converted into Brouncker's continued fraction $1 + 1^2/(2 + 3^2/(2 + 5^2/(2 + 7^2/(2 + \cdots))))$ for $4/\pi$. The author proves a simple condition for the irrationality of a continued fraction that can be applied to prove the irrationality of constants expressible by alternating series. His main result is that, if a series of type II is equivalent to a *simple* continued fraction, then the sum is transcendental and its irrationality measure exceeds 2. As a consequence, he reproves a result due to Davison and Shallit [DS91]: *Cahen's constant*

$$C = \sum_{n=0}^{\infty} \frac{(-1)^n}{S_n - 1} = 1 - \frac{1}{2} + \frac{1}{6} - \frac{1}{42} + \frac{1}{1806} - \cdots = 0.643410546288338\cdots$$

is transcendental, where $(S_n)_{n\geq 0} = (2, 3, 7, 43, 1807, 3263443, \ldots)$ is Sylvester's sequence, defined by the recursion $S_0 = 2$ and $S_{n+1} = (S_n - 1)S_n + 1$ for $n \geq 0$.

Around 1740, Euler discovered an explicit formula for the values of the zeta function at even integers, $\zeta(2n) = -\frac{(2\pi i)^{2n}}{2(2n)!} B_{2n}$, where B_n are the Bernoulli numbers defined by the generating series $\frac{z}{\exp(z)-1} = \sum_{n=0}^{\infty} B_n \frac{z^n}{n!}$ [Eul40]. Euler's formula shows that the positive critical values of the Riemann zeta function are nonzero rational multiples of powers of the period $2\pi i$. Together with Lindemann's theorem [Lin82], this implies the transcendence of all positive critical values of the Riemann zeta function. Hecke L-functions form a class of functions that are of great importance in number theory. They can be seen as a common generalization of the Riemann zeta function, Dirichlet L-functions and Dedekind zeta functions. In Chapter 23, Johannes Sprang summarizes the results on the algebraicity of critical Hecke L-values up to explicit periods and deduces (conditional) transcendence results for these critical L-values, obtained together with Guido Kings [KS20].

Alin Bostan
Kilian Raschel

[1] This chapter is published posthumously. We thank Wadim Zudilin for his suggestion to include late Jonathan Sondow's work in this volume.

General Description of the Conference

The main topics of the conference Transcendence in Transylvania were algebraic and transcendental aspects of special functions and special numbers arising in combinatorics, number theory, statistical mechanics and probability theory. The meeting was organized by Alin Bostan and Kilian Raschel (scientific committee), Nicusor Minculete, Eugen Păltănea and Diana Savin (local organizers). It was funded by the ERC grant “Elliptic Combinatorics” and by Inria. The Faculty of Mathematics and Computer Science of Transylvania University in Braşov provided useful support in the organization of this event.

About Braşov. Shadowed by mountains and containing a fine Baroque centre, Braşov is one of Transylvania’s most appealing cities. Its medieval old town is a marvelous introduction to the Saxon architecture of the region. Thanks to Bram Stoker and Hollywood, Transylvania (from the Latin for “beyond the forest”) is famed abroad as the homeland of Dracula, a mountainous place where storms lash medieval hamlets, while wolves—or werewolves—howl from the surrounding woods. But the Dracula image is just one element of Transylvania, whose near 100,000 square kilometres take in alpine meadows and peaks, caves and dense forests sheltering bears and wild boars and lowland valleys where buffalo cool off in the rivers. For the visitor, most striking of all are the stuhls, the former seats of Saxon power, with their medieval streets, defensive towers and fortified churches. One highlight of this corner is the castle at Bran (see the pictures on Fig. 2), which looks just how a vampire count’s castle should: a grim facade, perched high on a rock bluff, its turrets and ramparts rising in tiers against a dramatic mountain background. The Carpathian mountains are never far away in Transylvania, and for anyone fond of walking, this is one of the most beautiful while least exploited regions in Europe.

Fig. 2. Posters of the conference (Bran castle)

List of Participants

Seventy-one participants, from several countries and continents, attended the conference physically. David V. Chudnovsky and Gregory V. Chudnovsky gave a lecture by teleconference from New York, USA. The full list of participants is given below.

Fig. 3. Group photo of the conference

- Youssef Abdelaziz, Paris, France
- Axel Bacher, Paris, France
- Cristina Ballantine, Worcester, USA
- Nicholas Beaton, Melbourne, Australia
- Jason Bell, Waterloo, Canada
- Dan Betea, Bonn, Germany

- Frits Beukers, Utrecht, Netherlands
- Alin Bostan, Paris, France
- Kathrin Bringmann, Cologne, Gernmany
- Vasile Brînzănescu, Bucharest, Romania
- Xavier Caruso, Bordeaux, France
- David V. Chudnovsky, New York, USA
- Gregory V. Chudnovsky, New York, USA
- Frédéric Chyzak, Paris, France
- Alexandru Ciolan, Cologne, Germany
- Mihai Ciucu, Bloomington, USA
- Roxana Cojman, Bucharest, Romania
- Andrei Comăneci, Bucharest, Romania
- Éric Delaygue, Lyon, France
- Lucia Di Vizio, Paris, France
- Thomas Dreyfus, Strasbourg, France
- Andrew Elvey Price, Bordeaux, France
- Guy Fayolle, Paris, France
- Ştefan Garoiu, Braşov, Romania
- Tony Guttmann, Melbourne, Australia
- Charlotte Hardouin, Toulouse, France
- Herwig Hauser, Vienna, Austria
- Mark van Hoeij, Tallahassee, USA
- Mioara Joldes, Toulouse, France
- Frédéric Jouhet, Lyon, France
- Christoph Koutschan, Linz, Austria
- Christian Krattenthaler, Vienna, Austria
- Cédric Lecouvey, Tours, France
- Gabriel Lepetit, Grenoble, France
- Florian Luca, Johannesburg, South Africa
- Assia Mahboubi, Nantes, France
- Jean-Marie Maillard, Paris, France
- Marin Marin, Braşov, Romania
- Sergiu Megieşan, Cluj, Romania
- Mircea Merca, Craiova, Romania
- Preda Mihăilescu, Goettingen, Germany
- Nicuşor Minculete, Braşov, Romania
- Marni Mishna, Burnaby, Canad
- Vicenţiu Paşol, Bucharest, Romania
- Cristina Păcurar, Braşov, Romania
- Eugen Păltănea, Braşov, Romania
- Radu Păltănea, Braşov, Romania
- Carine Pivoteau, Marne-la-Vallée, France
- Alexandru Popa, Bucharest, Romania
- Dorin Popescu, Bucharest, Romania
- Cristian-Silviu Radu, Linz, Austria

- Kilian Raschel, Tours, France
- Tanguy Rivoal, Grenoble, France
- Julien Roques, Lyon, France
- Bruno Salvy, Lyon, France
- Diana Savin, Constanţa, Romania
- Michael F. Singer, Raleigh, USA
- Andrea Sportiello, Paris, France
- Johannes Sprang, Regensburg, Germany
- Alexandru Şerban, Braşov, Romania
- Alexandru Ştioboranu, Braşov, Romania
- Adrian Tanasa, Bordeaux, France
- Pierre Tarrago, Paris, France
- Stefan Trandafir, Burnaby, Canada
- George Turcas, Warwick, UK
- Brigitte Vallée, Caen, France
- Daniel Vargas, Lyon, France
- Bianca Vasian, Braşov, Romania
- Masha Vlasenko, Warsaw, Poland
- Michael Wallner, Bordeaux, France
- Jacques-Arthur Weil, Limoges, France
- Sergey Yurkevich, Vienna, Austria
- Alexandru Zaharescu, Urbana, USA

Talks and Abstracts

Semiabelian Varieties and Annihilators of Irreducible Representations

Jason Bell

Let G be a semiabelian variety defined over a field of characteristic zero and let $\Phi : G \rightarrow G$ be an endomorphism. We prove that if Y is a subvariety of G that intersects the orbit of every point of G that is periodic under the action of the map Φ, then Y is all of G. As a consequence of this result, we are able to give a topological characterization of the ideals that annihilate the irreducible representations in a class of algebras that "comes from geometry". This is joint work with Dragoş Ghioca.

Dwork's Congruences and the Cartier Operator

Frits Beukers

In his work on zeta functions for families of algebraic varieties, Dwork discovered a number of remarkable p-adic congruences. In this lecture, I discuss some joint work with Masha Vlasenko, in which we give a description of the action of the Cartier operator on the cohomology of the varieties and the resulting Dwork-type congruences. In this work, we only use elementary definitions and arguments.

False Theta Functions and Their Modular Properties

Kathrin Bringmann

In my talk, I will discuss modular properties of false theta functions. Due to a wrong sign factor, these are not directly seen to be modular; however, there are ways to repair this. I will report about this in my talk.

Algebraic Methods for Solutions of Bloch–Iserles Hamiltonian Systems

Vasile Brînzănescu

We shall present some results about the algebraic complete Hamiltonian systems of Bloch–Iserles. By using the spectral curve of such a system, we show that the solutions stay on a commutative algebraic group which is a non-trivial extension of a Prym variety (associated with the spectral curve and to an involution on it) by a multiplicative group $(\mathbb{C}^*)^s$. Finally, we try to give explicit solutions by θ-functions. This is a joint work with Cristina Maria Sandu.

2-adic Differential Equations and Isogenies

Xavier Caruso

Over the last decades, many algorithms were proposed to compute isogenies between elliptic curves. One of them is based on the observation that the rational function giving the isogeny is a solution of an explicit nonlinear differential equation of order 1. In characteristic zero, the latter equation has a unique solution which can be computed easily by a Newton iteration.

In this talk, I will focus on the case where the base field is a finite extension of field of 2-adic numbers $\mathbb{Q}_2$. This situation is particularly interesting because the differential equation we obtain in this case exhibits singularities close to 0, leading to huge numerical instability in the resolution step. In this talk, I will explain how Newton iteration can be modified in order to rub out most of the numerical stability and consequently highly improve the complexity of the isogeny computation.

This is a joint work with Elie Eid and Reynald Lercier.

Calculations of Classical Constants and Special Functions for Fun and Profit (Hardware View)

David V. and Gregory V. Chudnovsky

We describe challenges in contemporary high-performance computing, as exemplified by efforts of large-scale mathematical computations (e.g. π calculations) and of special function computations in applied area of mathematical finance and risk management.

Explicit Generating Series for Small-Step Walks in the Quarter Plane

Frédéric Chyzak

Lattice walks occur frequently in discrete mathematics, statistical physics, probability theory and operational research. The algebraic properties of their enumeration generating series vary greatly according to the family of admissible steps chosen to define them: their generating series are sometimes rational, algebraic, D-finite, differentially algebraic or sometimes they possess no apparent equation. This has recently motivated a large classification effort. Interestingly, the involved equations often have degrees, orders and sizes, making calculations an interesting challenge for computer algebra.

In this talk, we study nearest neighbours walks on the square lattice, that is, models of walks on the square lattice, defined by a fixed step set that is a subset of the 8 nonzero vectors with coordinates 0 or ± 1. We concern ourselves with the counting of walks constrained to remain in the quarter plane, counted by length. In the past, Bousquet–Mélou and Mishna showed that only 19 essentially different models of walks possess a non-algebraic D-finite generating series; the linear differential equations have then been guessed by Bostan and Kauers. In this work, we give the first proof that these equations are satisfied by the corresponding generating functions. This allows to derive nice formulas for the generating functions, expressed in terms of Gauss' hypergeometric series, to decide their algebraicity or transcendence. This also gives hope to extract asymptotic formulas for the number of walks counted by lengths.

(Based on joint work with Alin Bostan, Mark van Hoeij, Manuel Kauers and Lucien Pech.)

Lozenge Tilings of Doubly-Intruded Hexagons

Mihai Ciucu

Motivated in part by Propp's intruded Aztec diamond regions, we consider hexagonal regions out of which two horizontal chains of triangular holes (called ferns) are removed, so that the chains are at the same height and are attached to the boundary. In contrast to the intruded Aztec diamonds (whose number of domino tilings contain some large prime factors in their factorization), the number of lozenge tilings of our doubly intruded hexagons turns out to be given by simple product formulas in which all factors are linear in the parameters. We present in fact q-versions of these formulas, which enumerate the corresponding plane partitions-like structures by their volume. We also pose some natural statistical physics questions suggested by our set-up, which should be possible to tackle using our formulas. This is a joint work with Tri Lai.

Length Derivative of Generating Series for Walks in the Quarter Plane

Charlotte Hardouin

A walk in the quarter plane is a path in the square lattice starting at $(0,0)$ confined in the first quadrant that goes in each cardinal direction with a certain *probabilistic weight*. Its associated generating series is a trivariate formal power series of the form $Q(x,y,t) = \sum q_{i,j,k} x^i y^j t^k$ where $q_{i,j,k}$ is the probability for a walk to end at the point (i,j) in k steps. While the variables x and y are associated with the ending point of the walk, the variable t is associated with its length. In this talk, we study the algebraic differential equations satisfied by the generating series with respect to the t-derivation. We shall show how one can uniformize the problem in a non-Archimedean setting in order to associate with the generating series an auxiliary function satisfying a simple q-difference equation with meromorphic coefficients over a Tate curve. Then, difference Galois theory gives a dictionary between the differential algebraic relations of the series and the orbit configuration of a set of points on the elliptic curve associated with the Tate curve.

Methods from Commutative Algebra in the Study of Algebraic Power Series

Herwig Hauser

This lecture will be of expository type. We will look at various famous theorems about algebraic power series with the perspective of a commutative algebraist.

Factoring Linear Recurrence Operators

Mark van Hoeij

Several computer algebra systems have implementations for finding hypergeometric solutions of linear recurrence equations. This is equivalent to finding first-order factors of linear recurrence operators. This talk will present several approaches to compute higher-order factors of operators in $\mathbb{Q}(x)[\tau]$ where τ is the shift operator.

Enumeration of Diagonally Symmetric Alternating Sign Matrices

Christoph Koutschan

The number of $n \times n$ alternating sign matrices (ASMs) is given by a nice product formula that was conjectured by Mills, Robbins and Rumsey and that was proven by Zeilberger in 1992 and, shortly after, by Kuperberg. Also several symmetry classes of ASMs are enumerated by similar formulas. However, there are some classes which remain mysterious since their counting sequence appears incompatible with a product formula. We study one of these classes, namely ASMs that are symmetric with respect to the main diagonal and find a Pfaffian formula for their refined enumeration.

Elliptic Hypergeometric Series and Applications

Christian Krattenthaler

Hypergeometric series are ubiquitous in classical analysis and many other fields of mathematics. Their q-analogues, called basic hypergeometric series, underly the study of q-series and are invaluable tools in various problem areas of number theory and combinatorics. The theory of both—ordinary and basic hypergeometric series—has been largely developed in the first half of the twentieth century. A relatively recent development is the theory of elliptic hypergeometric series, which originates from work of Frenkel and Turaev on solutions of the Yang–Baxter equation from the end of the 1990s. These elliptic hypergeometric series contain both the ordinary and basic hypergeometric series as special cases. However, interestingly, work on these new series has (also) led to the discovery of identities that were even new when specialized to the ordinary or the basic case.

In my talk, I shall attempt to give an idea of the theory of these various hypergeometric series and then present several recent applications which concern problems in combinatorics, in special functions theory, respectively, in approximation theory.

X-Coordinates of Pell Equations In Various Sequences

Florian Luca

Let $d > 1$ be a square-free integer and (X_n, Y_n) be the nth solution of the Pell equation $X^2 - dY^2 = \pm 1$. Given your favourite set of positive integers U, one can ask what can we say about those d such that $X_n \in U$ for some n. Formulated in this way, the question has many solutions d, since one can always pick $u \in U$ and write $u^2 \pm 1 = dv^2$ with integers d and v such that d is square-free, obtaining in this way that (u, v) is a solution of the Pell equation corresponding to d. What about if we ask that $X_n \in U$ for at least two different ns? Then the answer is very different. For example, if U is the set of squares, then it is a classical result of Ljunggren that the only such d is 1785 for which both X_1 and X_2 are squares. In my talk, I will survey recent results about this problem when U is the set of Fibonacci numbers, Tribonacci numbers, k-generalized Fibonacci numbers, sums of two Fibonacci numbers, rep digits (in base 10 or any integer base $b \geq 2$) and factorials. The proofs use linear forms in logarithms and computations and in the case of factorials results about primes in arithmetic progressions. These results have been obtained in joint work with various colleagues such as J.J. Bravo, C. A. Gómez, S. Laishram, A. Montejano, L. Szalay and A. Togbé and recent Ph.D. students M. Ddamulira, B. Faye and M. Sias.

Truncated Theta Series, Partitions Inequalities and Rogers–Ramanujan Functions

Mircea Merca

My collaboration with George E. Andrews on the truncated version of Euler's pentagonal number theorem has opened up a new study on truncated theta series. Since then, over twenty papers on this topic have followed, and several partition inequalities are derived in this way. In this talk, we present a very general method for proving the non-trivial linear homogeneous partition inequalities. This method does not involve truncated theta series or q-series and connects the non-trivial linear homogeneous partition inequalities with the Prouhet–Tarry–Escott problem. On the other hand, I present an improvement of a conjecture related to a truncated theta series which I gave with G. E. Andrews in 2012. Combinatorial interpretations of this new conjecture give, for each $S \in \{1, 2, 3, 4\}$, an infinite family of linear homogeneous inequalities for the number of partitions of n into parts congruent to $\pm S \bmod 5$. Twenty new identities involving the Rogers–Ramanujan functions $G(q)$ and $H(q)$ are experimentally discovered in this way.

The Charm of Units—The Conjecture of Kummer and Vandiver

Preda Mihăilescu

A classical conjecture of elementary algebraic number theory—which can be stated in a short phrase involving well-known concepts—but yet unsolved since more than a century, is the Kummer–Vandiver conjecture. It states that the p-part of the class group of the maximal real cyclotomic p-th field—thus the extension of $\mathbb{Q}[\zeta + \bar{\zeta}]$ of $\mathbb{Q}$ by the sum of a p-th root of unity ζ and its complex conjugate—is trivial. I solve this conjecture in two steps: the first, more unexpected one, will be presented in a main session—it was first presented in 2017 in some conferences in India and China, then in France and appears in the proceedings of one of these events. It states that the conjecture is proved, provided that a related, asymptotic conjecture due to Ralf Greenberg, is true. The second step will be presented in an off-schedule seminary, which we offer for those interested to see the result in more detail. It contains, of course, the proof of the Greenberg conjecture, for this p-th cyclotomic field.

A Combinatorial Refinement of the Kronecker–Hurwitz Class Number Relation

Alexandru Popa

I will present a refinement, and a new proof, of the classical Kronecker–Hurwitz class number relation, based on a tessellation of the Euclidean plane into semi-infinite triangles labelled by the modular group. This is a joint work with Don Zagier.

The Bass–Quillen Conjecture and Swan's Question

Dorin Popescu

In 1982, R. Swan noticed that, for an answer to the Bass–Quillen conjecture, it would be useful to prove that any regular local ring (R, m, k) is a filtered inductive limit of regular local rings, essentially of finite type over $\mathbb{Z}$. In 1989, we gave a positive answer to the Swan's question when $p = \mathrm{char}(k)$ is either zero, or $p \notin m^2$, or $0 \neq p \in m^2$ but R is excellent Henselian. In all these questions R is excellent. Last year, Kęstutis Česnavičius wanted to know that in general a regular local ring is excellent because this will allow him to reduce the purity conjecture to the case of regular local rings which are complete. So we had to give a complete positive answer to Swan's question.

Linear Independence of Values of *G*-Functions

Tanguy Rivoal

G-functions are holomorphic functions at 0 solutions of linear differential equations with polynomial coefficients, and whose algebraic Taylor coefficients at 0 satisfy certain growth conditions. They were defined and studied by Siegel in 1929, and they can be viewed as generalizations of $\log(1-z)$: they include for instance Gauss hypergeometric series with rational parameters, polylogarithms, algebraic functions, derivatives and primitives of such functions. I will first present classical Diophantine results concerning the values taken by a *G*-function and its derivatives at algebraic points close to 0 (Siegel, Galochkin, Chudnovsky). I will then present a new Diophantine result concerning the dimension of the vector space generated over $\mathbb{Q}$ by the values taken by a *G*-function and (essentially) its primitives at any algebraic point inside their disc of convergence. This is a joint work with Stéphane Fischler.

Finite Automata, Automatic Sets and Difference Equations

Michael F. Singer

A finite automaton is one of the simplest models of computation. Initially introduced by McCulloch and Pitts to model neural networks, they have been used to aid in software design as well as to characterize certain formal languages and number-theoretic properties of integers. A set of integers is said to be m-automatic if there is a finite automaton that decides if an integer is in this set given its base-m representation. For example, powers of 2 are 2-automatic but not 3-automatic. This latter result follows from a theorem of Cobham describing in which sets of integers are m- and n-automatic for sufficiently distinct m and n. In recent work with Reinhard Schaefke, we gave a new proof of this result based on analytic results concerning normal forms of systems of difference equations. In this talk, I will describe this circle of ideas.

The Tangent Method for the Determination of Arctic Curves

Andrea Sportiello

In the paper [CS16] of Filippo Colomo and myself, we pose the basis for a method aimed at the determination of the "arctic curve" of large random combinatorial structures, i.e. the boundary between regions with zero and nonzero local entropy, in the scaling limit. This "basic" version of the tangent method (TM) is strikingly simple, but unfortunately it is not completely rigorous.

Two other versions of the method exist, let us call them the "entropic" tangent method (E-TM) and the "double-refinement" tangent method (2R-TM). In this talk, we shall first briefly review the "basic" TM, then we will introduce the two other methods and explain how the 2R-TM is completely rigorous, but it involves more complex quantities, while the E-TM has essentially the same technical difficulties of the TM, but it is even more heuristic. Finally, we close the circle, by showing how the Desnanot–Jacobi identity applied to the Izergin determinant implies the equivalence between the E-TM and the 2R-TM in the case of the six-vertex model with domain wall boundary conditions.

The Jacobian Conjecture, a Reduction of the Degree via a Combinatorial Physics Approach

Adrian Tanasa

The Jacobian conjecture is a celebrated conjecture stating (since 1939!) that any locally invertible polynomial system in $\mathbb{C}^n$ is globally invertible with polynomial inverse. C. W. Bass et al. (1982) proved a reduction theorem stating that the conjecture is true for any degree of the polynomial system if it is true in degree three. This degree reduction is obtained with the price of increasing the dimension n. I will show in this talk a theorem concerning partial elimination of variables, which implies a reduction of the generic case to the quadratic one. The price to pay is the introduction of a supplementary parameter $0 < n' < n$, parameter which represents the dimension of a linear subspace where some particular conditions on the system must hold. This result was obtained using the so-called intermediate field method in a quantum field theoretical (QFT) reformulation of the Jacobian conjecture. I will first present the general idea of this QFT method and then show how it applies to obtain our reduction result for the Jacobian conjecture.

Degeneration of Frobenius Structures

Masha Vlasenko

It was observed by Dwork that matrices of p-adic Frobenius operators in families of algebraic varieties satisfy differential equations. This led to the notion of Frobenius structures. We study degeneration of Frobenius structures at a singular point. As numerical evaluation shows, entries of degenerate Frobenius matrices may contain special values of p-adic L-functions. Most of our examples will be families of hypersurfaces with hypergeometric Picard–Fuchs equations. We will discuss a relevant conjecture of Candelas, de la Ossa and van Straten.

Zeros of the Riemann Zeta Function on the Critical Line

Alexandru Zaharescu

We discuss some recent developments that allow one to conclude that more than 5/12 of the non-trivial zeros of the Riemann zeta function lie on the critical line.

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