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Handbook of Geometry and Topology of Singularities III

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Preface

This is the third volume of the *Handbook of the Geometry and Topology of Singularities*, a subject which is ubiquitous in mathematics, appearing naturally in a wide range of different areas of knowledge. The scope of singularity theory is vast, its purpose is multifold. This is a meeting point where many areas of mathematics, and science in general, come together.

Let us reminisce Bernard Teissier's words in his foreword to Volume I of the Handbook:

> I claim that Singularity Theory sits inside Mathematics much as Mathematics sits inside the general scientific culture. The general mathematical culture knows about the existence of Morse theory, parametrizations of curves, Bézout's theorem for plane projective curves, zeroes of vector fields and the Poincaré-Hopf theorem, catastrophe theory, sometimes a version of resolution of singularities, the existence of an entire world of commutative algebra, etc. But again, for the singularist, these and many others are lineaments of a single landscape and she or he is aware of its connectedness. Moreover, just as Mathematics does with science in general, singularity theory interacts energetically with the rest of Mathematics, if only because the closures of non singular varieties in some ambient space or their projections to smaller dimensional spaces tend to present singularities, smooth functions on a compact manifold must have critical points, etc. But singularity theory is also, again in a role played by Mathematics in general science, a crucible where different types of mathematical problems interact and surprising connections are born.

The Handbook has the intention of covering a wide scope of singularity theory, presenting articles on various aspects of the theory and its interactions with other areas of mathematics in a reader-friendly way. The authors are world experts; the various articles deal with both classical material and modern developments.

The first Volume I of this collection gathered ten articles concerning foundational aspects of the theory. This includes:

- The combinatorics and topology of plane curves and surface singularities
- An introduction to four of the classical methods for studying the topology and geometry of singular spaces, namely resolution of singularities, deformation theory, Stratifications, and slicing the spaces *à la* Lefschetz
- Milnor fibrations and their monodromy

– Morse theory for stratified spaces and constructible sheaves
– Simple Lie algebras and simple singularities

Volume II also consists of ten articles. These cover foundational aspects of the theory as well as some important relations with other areas of mathematics. They include:

– The analytic classification of plane curve singularities and the existence of complex and real algebraic curves in the plane with prescribed singularities
– An introduction on the limits of tangents to a complex analytic surface, a subject that originates in Whitney's work
– Introductions to Zariski's equisingularity and intersection homology, which are two of the main current viewpoints for studying singularities
– An overview of Milnor's fibration theorem for real and complex singularities, as well as an introduction to Massey's theory of Lê cycles
– A discussion of mixed singularities, which are real analytic singularities with a rich structure that allows their study via complex geometry
– The study of intersections of concentric ellipsoids in $\mathbb{R}^n$ and its relation with several areas of mathematics, from holomorphic vector fields to singularity theory, toric varieties, and moment-angle manifolds
– A review of the topology of quasi-projective varieties and generalizations about the complements of plane curves and hypersurfaces in projective space

This Volume III also consists of ten chapters. Some of these complement topics explored previously in Volumes I and II, while other chapters bring in important new subjects. Let us say a few words about the content of this volume, though each chapter has its own abstract, introduction and a large bibliography for further reading. There is also a global index of terms at the end.

Chapters 1 and 2 have as common thread the much celebrated Thom-Mather theory. In 1944, Whitney studied mappings $\mathbb{R}^n \to \mathbb{R}^{2n-1}$, the first pair of dimensions not covered by his immersion theorem, showing that in these setting singularities cannot be avoided in general. He then introduced the concept of stable mappings and characterized the stable mappings from $\mathbb{R}^n$ to $\mathbb{R}^p$ with $p \geq 2n-1$, and also those from the plane into itself, showing that in all these cases the stable mappings form a dense set in the space of smooth proper mappings. Whitney conjectured that the density of stable mappings would hold for any pair (n, p). However, René Thom showed that this is not the case by giving a counterexample. Thom then conjectured that the topologically stable maps are always dense and gave an outline of the proof. The complete proof was given by John Mather, who, from 1965 to 1975, solved almost completely the program drawn by Thom for the stability problem. This is known as Thom-Mather theory.

Chapter 3 is about Zariski's equisingularity, previously envisaged in Parusiński's chapter in Volume II. Among the various notions of equivalence of singularities, topological equisingularity is one of the oldest and easiest to define, but it is far from being well understood. Several challenging questions remain open. In this chapter, the author surveys developments in topological equisingularity, some of its relations with other equisingularity notions, and hints on new possible approaches to old questions based in algebro-geometric methods, Floer theory, and Lipschitz geometry. Topological equisingularity questions were crucial motivation sources for the development of the Computer Algebra program SINGULAR; this is explained in an appendix by G.-M. Greuel and G. Pfister.

Chapter 4 somehow fits within the classical interplay between normal singularities in complex surfaces and 3-manifold theory, which has been studied for decades and was discussed from a topological perspective in F. Michel's chapter in Volume I. Now the author looks at the subject from another perspective, bringing in subtle structures. Given a complex analytic normal surface singularity $(X, 0)$ we know that its topology is fully determined by its link L_X, a 3-manifold which is the intersection of X with a sufficiently small sphere in the ambient space, centered at 0. The main motif of this chapter is studying the ties between analytic and topological invariants of $(X, 0)$. Historically, this program was started by Artin and Laufer, which characterized topologically the rational and minimally elliptic singularities (respectively), and computed several analytic invariants from the resolution graph. This question brings us into the theory of the Casson and Casson-Walker invariant, the (refined) Turaev torsion, Seiberg-Witten invariants, lattice (co)homology, Heegaard-Floer theory, and other important invariants of 3-manifolds. This chapter starts from well-known elementary facts about surface singularities and brings us to the depths of this rich and interesting theory.

Chapters 5–7 discuss different aspects of the theory of Chern classes for singular varieties. For complex manifolds, their Chern classes are by definition those of its tangent bundle. These are important invariants that encode deep geometric and topological information. When we consider singular varieties, there is not a unique way of extending this concept. This somehow depends on which properties of Chern classes we are interested in, or how we extend the notion of the tangent bundle over the singular set. In these chapters, the authors introduce in elementary ways the various notions of Chern classes for singular varieties and their relations with other invariants of singular varieties. Chapter 5 gives a thorough account of the subject, from the birth of the theory of Chern classes up to the modern theories of motivic, bivariant, and Hirzebruch characteristic classes. Chapter 6 has Segre classes as its core. These classes are an important ingredient in Fulton-MacPherson intersection theory and provide a powerful mean for studying Chern classes of vector bundles in the algebraic setting. Several important invariants of algebraic varieties may be expressed in terms of Segre classes. The goal of that chapter is to survey several invariants specifically arising in singularity theory which may be defined or recast in terms of Segre classes. Chapter 7 looks at the subject from a topological viewpoint, focusing on the relations between local and global invariants, particularly indices of vector fields, the Milnor number, and Lê cycles. It includes for completeness an introduction to the Hirzebruch-Riemann-Roch theorem and its generalizations to singular varieties that give rise to several of the recent developments in the subject.

Chapter 8 studies the residues in complex analytic varieties that arise from the localization of characteristic classes via Alexander duality. A paradigm for this theory is the classical theorem of Poincaré-Hopf that can be understood as providing a localization of the top Chern class of a complex manifold at the singularities of a vector field. This was beautifully extended by Baum and Bott for singular holomorphic foliations on complex manifolds, providing expressions for certain Chern numbers in terms of residues localized at the singular set of the foliation. The theory that the author presents in this chapter starts with the study of residues of singular holomorphic foliations, later transferred to the index theory of holomorphic self-maps. The philosophy behind is rather simple. Namely, once we have some kind of vanishing theorem on the non-singular part of a geometric object such as a foliation, certain characteristic classes are localized at the set of singular points, and the localization gives rise to residues via the Alexander duality. The author explains how the relative Čech-de Rham theorem allows us to deal with the problem from both the topological and differential geometric viewpoints, and the comparison of the two yields various interesting expressions of the residues and applications.

Chapter 9 surveys applications of mixed Hodge theory to the study of isolated singularities. Hodge theory deals with the cohomology of smooth complex projective varieties, or more generally, compact Kähler manifolds. A choice of a Riemannian metric enables one to define the Laplace operator Δ on differential forms, and each de Rham cohomology class contains exactly one closed form ω with $\Delta\omega = 0$, the harmonic representative. One has the Hodge decomposition of cohomology classes via their harmonic representatives:

$$H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X)$$

where $H^{p,q}(X)$ is the subspace of $H^k(X, \mathbb{C})$ consisting of classes of forms containing harmonic forms of type (p, q). Using Leray's theory of sheaves and resolution of singularities, Grothendieck defined the de Rham cohomology of complex algebraic varieties in purely algebraic terms. A generalization of Hodge theory to arbitrary complex algebraic varieties was then developed by Deligne. He showed that the cohomology of a complex algebraic variety carries a slightly more general structure, which presents $H^k(X, \mathbb{C})$ as a successive extension of Hodge structures of decreasing weights. This generalization is called a mixed Hodge structure.

We close this volume with Chap. 10, a detailed introduction of the theory of constructible sheaf complexes in the complex algebraic and analytic settings. All concepts are illustrated by many interesting examples and relevant applications, while some important results are presented with complete proofs. This chapter is intended as a broadly accessible user's guide to those topics, providing the readers not only with a presentation of the subject but also with concrete examples and applications that motivate the general theory. The authors introduce the main results of stratified Morse theory in the framework of constructible sheaves, a subject discussed also in Goresky's chapter in Volume I of this Handbook. Constructible sheaf complexes and especially perverse sheaves have become indispensable tools for studying complex algebraic and analytic varieties. They have seen spectacular applications in geometry and topology, and several of these are discussed in this chapter.

This handbook is addressed to graduate students and newcomers to the theory, as well as to specialists who can use it as a guidebook. It provides an accessible account of the state of the art in several aspects of the subject, its frontiers, and its interactions with other areas of research. This will continue with a Volume IV, which will cover other aspects of singularity theory, and a Volume V, which will focus on holomorphic foliations, a remarkably important subject on its own that has close connections with singularity theory.

We thank Bernard Teissier for allowing us to use his words above and for valuable and inspiring comments.

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