José Luis Cisneros Molina
Dũng Tráng Lê
José Seade *Editors*

Handbook of Geometry and Topology of Singularities I

Springer

Handbook of Geometry and Topology of Singularities I

José Luis Cisneros Molina • Dũng Tráng Lê •
José Seade
Editors

Handbook of Geometry and Topology of Singularities I

Springer

Editors
José Luis Cisneros Molina
Unidad Cuernavaca
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Cuernavaca, Mexico

Unidad Mixta Internacional 2001 CNRS
Laboratorio Solomon Lefschetz
Cuernavaca, Mexico

José Seade
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Mexico City, Mexico

Unidad Mixta Internacional 2001 CNRS
Laboratorio Solomon Lefschetz
Cuernavaca, Mexico

Dũng Tráng Lê
Centre de Mathématiques et Informatique
Université d'Aix-Marseille
Marseille, France

ISBN 978-3-030-53060-0 ISBN 978-3-030-53061-7 (eBook)
https://doi.org/10.1007/978-3-030-53061-7

Mathematics Subject Classification: M11019, M11132, M12198, M21022, M21050, M28027,17B45

This Springer imprint is published by the registered company Springer Nature Switzerland AG.
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Foreword

In the general scientific culture, Mathematics can appear as quite disconnected. One knows about calculus, complex numbers, Fermat's last theorem, convex optimization, fractals, vector fields and dynamical systems, the law of large numbers, projective geometry, vector bundles, the Fourier transform and wavelets, the stationary phase method, numerical solutions of PDEs, etc., but no connection between them is readily apparent. For the mathematician, however, all these and many others are lineaments of a single landscape. Although he or she may spend most of his or her time studying one area of this landscape, the mathematician is conscious of the possibility of traveling to other places, perhaps at the price of much effort, and bringing back fertile ideas. Some of the results or proofs most appreciated by mathematicians are the result of such fertilizations.

I claim that Singularity Theory sits inside Mathematics much as Mathematics sits inside the general scientific culture. The general mathematical culture knows about the existence of Morse theory, parametrizations of curves, Bézout's theorem for plane projective curves, zeroes of vector fields and the Poincaré–Hopf theorem, catastrophe theory, sometimes a version of resolution of singularities, the existence of an entire world of commutative algebra, etc. But again, for the singularist, these and many others are lineaments of a single landscape and he or she is aware of its connectedness. Moreover, just as Mathematics does with science in general, singularity theory interacts energetically with the rest of Mathematics, if only because the closures of non-singular varieties in some ambient space or their projections to smaller dimensional spaces tend to present singularities, smooth functions on a compact manifold must have critical points, etc. But singularity theory is also, again in a role played by Mathematics in general science, a crucible where different types of mathematical problems interact and surprising connections are born.

- Who would have thought in the 1950s that there was a close connection between the classification of differentiable structures on topological spheres and the boundaries of certain isolated singularities of complex hypersurfaces?

- or that Thom's study of singularities of differentiable mappings would give birth to a geometric vision of bifurcation phenomena and of fundamental concepts such as structural stability?
- Who would have thought in the 1970s that there was a relation between the work of Lefschetz comparing the topological invariants of a complex projective variety with those of a general hyperplane section and the characterization of the sequences of integers counting the numbers of faces of all dimensions of simple polytopes?
- Or that one could produce real projective plane curves with a prescribed topology by deforming piecewise linear curves in the real plane?
- Or in the 2000s that properties of the intersections of two curves on a complex surface would lead to the solution of a problem connected with the coloring of graphs?
- Or that the algebraic study of the space of arcs on the simplest singularities ($z^n = 0$ in $\mathbf{C}$, $n \geq 2$) would provide new proofs and generalizations of the Rogers–Ramanujan and Gordon identities between the generating series of certain types of partitions of integers?

These are only a few examples. But to come back to the theory of singularities, I would like to emphasize that what I like so much about it is that not only are surprising connections born there but also very simple questions lead to ideas which resonate in other part of the field or in other fields. For example, if an analytic function has a small modulus at a point, does it have a zero at a distance from that point which is bounded in terms of that modulus? what is a general smooth function on a smooth compact manifold? A Morse function, with very mild singularities! And what happens if you replace the smooth manifold by a space with singularities? And then, given a function, can we measure how far it is from behaving like a general function? Suppose that a holomorphic function has a critical point at the origin. How can we relate the nature of the fiber of the function through this critical point with the geometry or topology of the nearby non-singular fibers? How can we relate it with the geometry of the mappings resolving singularities of this singular fiber? Then again, what is a general map between smooth manifolds? and how do you deform a singular space into a non-singular one in general? Well, that is more complicated. But I hope you get the idea.
The downside is that before he or she can successfully detect and try to answer such apparently simple and natural questions, the student of singularities must become familiar with different subjects and their techniques, and the learning process is long.

And this is why a handbook which presents in-depth and reader-friendly surveys of topics of singularity theory, with a carefully crafted preface explaining their place within the theory, is so useful!

Paris, France
March 2020

Bernard Teissier

Preface

Singularities are ubiquitous in mathematics, appearing naturally in a wide range of different areas of knowledge. They are a meeting point where many areas of mathematics and science in general come together. Their scope is vast, their purpose is multifold.

Singularity theory dates back to I. Newton, É. Bézout, V. Puiseux, F. Klein, M. Noether, F. Severi, and many others. Yet, it emerged as a field of mathematics in itself in the early 1960s, thanks to pioneering work by R. Thom, O. Zariski, H. Whitney, H. Hironaka, J. Milnor, E. Brieskorn, C. T. C. Wall, V. I. Arnold, J. Mather, and many others. Its potential for applications in other areas of mathematics and of knowledge in general is unlimited, and so are its possible sources of inspiration.

As the name suggests, one may naively say that singularity theory studies that which is "singular," that which is different from "most of the rest," different from its surroundings. As basic examples, we may look at the critical points of smooth functions, or at the points where a space loses its manifold structure, at the stationary points of flows and the special orbits of Lie group actions, at bifurcation theory and properties of objects or situations depending on parameters that undergo sudden change under a small variation of the parameters. These are some examples, out of a myriad of possibilities, of how singularities arise. There is great richness in the subject, and the literature is vast, with plenty of different viewpoints, perspectives, and interactions with other areas. That makes this subject fascinating.

That same wideness and amplitude of its scope can make singularity theory hard to grasp for graduate students and researchers in general: what are and what have been the major lines of development in the last decades, what is known and where to find it, what is the current state of the art in its many branches, the various directions into which this theory is flourishing, its interaction with other areas of current research in mathematics. Those are questions that gave birth to this project, the "Handbook of Geometry and Topology of Singularities."

This handbook has the intention of covering a wide scope of singularity theory, presenting articles on various aspects of the theory and its interactions with other areas of mathematics. The authors are world experts; the various articles deal with both classical material and modern developments. They are addressed to graduate students and newcomers into the theory, as well as to specialists that can use these as guidebooks.

Volume I consists of ten articles that cover some of the foundational aspects of the theory. This includes:

- The combinatorics and topology of plane curves and surface singularities.
- An introduction to four classical methods for studying the topology and geometry of singular spaces, namely: resolution of singularities, deformation theory, stratifications, and slicing the spaces *à la* Lefschetz.
- Milnor fibrations and their monodromy.
- Morse theory for stratified spaces and constructible sheaves.
- Simple Lie algebras and simple singularities.

We say below a few words about the content of each chapter. Of course, due to lack of space, many important topics from the geometric study of singularities are missing from this volume. This will be compensated to some extent in the next volumes. Also, the number of possible authors much exceeds the capacity of any project of this kind. We thank our many colleagues that have much contributed to build up singularity theory, and we apologize for our omissions in the selection of subjects. Among the topics we plan to include in later volumes of this Handbook of Geometry and Topology of Singularities are:

- Equisingularity.
- Lipschitz geometry in singularity theory.
- The topology of the complement of arrangements and hypersurface singularities.
- Mixed Hodge structures.
- Analytic classification of singularities of complex plane curves.
- Applications to Lagrangian and Legendrian geometry.
- Contact and symplectic geometry in singularity theory.
- Indices of vector fields and 1-forms on singular varieties.
- Chern classes of singular varieties.
- Tropical geometry and singularity theory.
- Milnor fibrations for real analytic maps.
- Mixed singularities.
- Singularities of map germs. Finite determinacy and unfoldings.
- Relations with moment angle manifolds.
- Invariant algebraic sets in holomorphic dynamics.
- Limits of tangent spaces.
- Invariants of 3-manifolds and surface singularities.
- Zeta functions and the monodromy.

Chapters 1 and 2 of this volume deal with dimensions 1 and 2, respectively. Chapter 1, by Evelia García Barroso, Pedro González Pérez, and Patrick Popescu-Pampu, is entitled “The Combinatorics of Plane Curve Singularities: How Newton Polygons Blossom into Lotuses.” In this chapter, the authors discuss classical ways to describe the combinatorics of singularities of complex algebraic curves contained in a smooth complex algebraic surface. In fact, given a smooth complex surface S

and a complex curve C in S with a singular point o, it is customary to study the local structure of (S, C) near o in the following ways:

- By choosing a local parametrization of C. This method dates back to Newton and later Puiseux. The combinatorics in it may be encoded in the Kuo-Lu tree and a Galois quotient of it, the Eggers-Wall tree.
- By blowing up points to obtain an embedded resolution of C. This blow-up process may be encoded in an Enriques diagram and a corresponding weighted dual graph.
- By performing a sequence of toric modifications. The combinatorial data generated during this process can be encoded in a sequence of Newton polygons and Newton fans.
- By looking at the intersection of S and C with a small sphere in some ambient space $\mathbb{C}^n$. One gets a knot (or link) in a 3-sphere. These are all iterated torus knots known as algebraic knots. Their combinatorics is encoded in the Puiseux pairs.

Chapter 1 studies the first three of these methods and explains how the notion of lotus, which is a special type of simplicial complex of dimension 2, allows to think simultaneously about the combinatorics of those three ways of analyzing the curve singularity.

The fourth method mentioned above is actually much related to Chap. 2 in this volume, by Françoise Michel, entitled "The Topology of Surface Singularities." This chapter surveys the subject of the topology of complex surface singularities. This classical subject dates back to Felix Klein and his work on invariant polynomials for the finite subgroups of the special unitary group SU(2). This gave rise to what today are called Klein singularities, though they have many names, as, for instance, Du Val singularities, rational double points, and simple singularities in Arnold's classification. If X is a complex surface singularity with base point p in some ambient space $\mathbb{C}^n$, then the intersection $L_X = X \cap \mathbb{S}_e$ with a small sphere centered at p is a 3-dimensional real analytic variety, whose topology is independent of the choice of the embedding of X in $\mathbb{C}^n$ and also independent of the choice of the (sufficiently small) sphere; L_X is called the link of the singularity and it fully describes the topology of X. If X has an isolated singularity at p, then L_X is a 3-manifold. The manifolds one gets in this way are all Waldhausen (or graph) manifolds that can be constructed by plumbing, a technique introduced by John Milnor in all dimensions, in order to construct the first examples of homology spheres. The author also gives an explicit construction of a good resolution of the singularity, and the minimal good resolution by the Hirzebruch–Jung method is described in detail.

Chapters 3 to 9 deal with the four classical ways of studying the geometry and topology of singular spaces mentioned above, namely:

1. Via resolutions of the singularities;
2. Via stratifications;

3. Via deformations, smoothings, and unfoldings; and
4. Taking slices with the fibers of a linear form.

Let us say a few words about each of them.

The problem of resolution of singularities and its solution in various contexts, already discussed for plane curves in Chap. 1 and for surfaces in Chap. 2, can be traced back to Newton and Riemann. Chapter 3, by Mark Spivakovsky, is an introduction to the resolution of singularities. This surveys the subject, starting with Newton till the modern times. It also discusses some of the main open problems that remain to be solved. The main topics covered are the early days of the subject, Zariski's approach via valuations, Hironaka's celebrated result in characteristic zero and all dimensions and its subsequent strengthenings and simplifications, existing results in positive characteristic (mostly up to dimension three), de Jong's approach via semi-stable reduction, Nash and higher Nash blowing up, as well as reduction of singularities of vector field and foliations.

Chapter 4 is an introduction to the stratification theory, by David Trotman. The idea behind the notion of stratification in differential topology and algebraic geometry is to partition a (possibly singular) space into smooth manifolds with some control on how these manifolds fit together. In 1957, Whitney showed that every real algebraic variety V in $\mathbb{R}^n$ can be partitioned into finitely many connected smooth submanifolds of $\mathbb{R}^n$. This he called a manifold collection. In 1960, René Thom replaced the term manifold collection by stratified set and initiated a theory of stratified sets and stratified maps. In this chapter, the author presents in a unifying manner both the abstract theory of stratified sets elaborated by Thom, Whitney, and Mather and the stratification theory of semi-algebraic, subanalytic, or complex analytic sets. In addition, it surveys the relations between several stratifying conditions which are modifications of the Whitney conditions, with an emphasis on the applications to the openness of transversality theorems which are so important in stability problems. The text also explains what remains true of the stratification theory of real algebraic and subanalytic sets in the o-minimal framework.

Chapter 5 by Mark Goresky, entitled "Morse Theory, Stratifications and Sheaves," begins with an introduction to Morse theory for stratified spaces and then moves forward to discussing how stratified Morse theory and the theory of constructible sheaves, introduced by M. Kashiwara and P. Shapira, are two sides of the same coin. A complete and parallel development of the two theories was presented by J. Schürmann. In this chapter, the author provides an intuitive view of this parallel development. The setting presented by Schürmann replaces the subanalytic and Whitney stratified setting with the more general conditions of o-minimal structures and generalized Whitney conditions: w-regularity, d-regularity, and C-regularity. In this chapter, for simplicity, the author remains within the subanalytic and Whitney stratified setting.

Chapter 6, by J. J. Nuño Ballesteros, Lê D. T., and J. Seade, treats a now classical and central subject in singularity theory: the Milnor fibration theorem, which provides the simplest example of a deformation of a singular variety into a smooth one. This fibration theorem, published by John Milnor in 1968, concerns the geometry and topology of analytic maps near their critical points, and it was the culmination of a series of articles by Brieskorn, Hirzebruch, Pham, and others, aimed toward finding complex isolated hypersurface singularities whose link, i.e., its intersection with a small sphere centered at the singular point, is a homotopy sphere. The theorem considers a nonconstant holomorphic map germ $(\mathbb{C}^{n+1}, \underline{0}) \xrightarrow{f} (\mathbb{C}, 0)$ with a critical point at $\underline{0}$, and it can roughly be stated as saying that the local noncritical levels $f^{-1}(t)$ form a locally trivial C^∞ fiber bundle over a sufficiently small punctured disc in $\mathbb{C}$. Notice that one has a flat family F_t of complex manifolds degenerating to the special fiber $f^{-1}(0)$. This is the paradigm of a smoothing, i.e., a flat deformation where all fibers, other than the special one, are non-singular. Milnor's fibration theorem is a cornerstone in singularity theory. It has opened several research fields and given rise to a vast literature. In this chapter, the authors present some of the foundational results about this subject and give proofs of several basic "folklore theorems" which either are not in the literature or are difficult to find. They also glance at the use of polar varieties, developed by Lê and Teissier, for studying the topology of singularities. This springs from ideas by René Thom and relates to the subject mentioned above, of studying singular varieties by slicing them by the fibers of a linear form. The chapter includes a proof of the "attaching-handles" theorem, which is key for Lê–Perron and Massey's theory describing the topology of the Milnor fiber. It also discusses the so-called carousel that allows a deeper understanding of the topology of plane curves (as in Chap. 1) and has several applications in various settings. Finally, two classical open problems in complex dimension two are discussed: Lê's conjecture and the Lê–Ramanujam problem.

Deformation theory, together with the resolution of singularities and stratifications, is one of the fundamental methods for the investigation of singularities. In Chap. 7, entitled, "Deformation and Smoothing of Singularities," Gert-Martin Greuel gives a comprehensive survey of the theory of deformations of isolated singularities and the related question of smoothability. The basic general theory is systematically and carefully presented and the state of the art corresponding to the most important questions is exhaustively discussed. The article contains almost no proofs, but references to the relevant literature, in particular to the textbook of Greuel, Lossen, and Shustin "Introduction to Singularities and Deformations." As in this book, there are some examples treated with Singular, a computer algebra system for polynomial computations. Relations are given between different invariants, such as the Milnor number, the Tjurina number, and the dimension of a smoothing component.

Chapter 8, by Wolfgang Ebeling, gives an introduction to "Distinguished Bases and Monodromy of Complex Hypersurface Singularities," a fundamental topic for understanding the Milnor fibration. The Milnor fibration essentially is a fiber bundle over the circle S^1. Therefore, it is determined by the fiber and by the monodromy map: if we think of S^1 as being obtained from the interval [0, 1] by gluing its end points, then the (geometric) monodromy is a diffeomorphism from the fiber over {0} to that over {1}, telling us how to glue the fibers in order to recover the original bundle. In the isolated singularity case, the fiber F_t (which is the local noncritical level) has the homotopy type of a bouquet of spheres of middle dimension n; the number of such spheres is the aforementioned Milnor number μ. Hence all reduced homology groups of F_t vanish, except $H_n(F)$ which is free abelian of rank μ. The elements in $H_n(F)$ are called vanishing cycles. The geometric monodromy induces an automorphism of $H_n(F)$, known as the monodromy of the map germ f. A natural way to study the monodromy operator is by finding "good" bases for $H_n(F; \mathbb{Z}) \cong \mathbb{Z}^\mu$. Such a concept was made precise by Gabrielov in the 1970s, introducing the notion of "distinguished bases." These fundamental concepts and their further developments are discussed in Chap. 8.

One of the basic problems of algebraic geometry is to extract topological information from the equations which define an algebraic variety. The theorem of Lefschetz for hyperplane sections shows that when the base field is the field of complex numbers and the projective variety is non-singular, one can, to some extent, compare the topology of a given projective variety with that of a hyperplane section. In Chap. 9, "Lefschetz Theorem for Hyperplane Sections," by Helmut Hamm and Lê Dũng Tráng, the authors consider different theorems of Lefschetz type. The chapter begins with the classical Lefschetz hyperplane sections theorem on a non-singular projective variety. Then they show that this extends to the cases of a non-singular quasi-projective variety and to singular varieties. They also consider local forms of theorems of Lefschetz type.

As mentioned earlier in this introduction in relation with Chap. 2, Felix Klein studied the action of the finite subgroups G of SU(2) on the complex space $\mathbb{C}^2$ that give rise to the surface singularities $\mathbb{C}^2/G$, which are known nowadays as Klein singularities. Later, in the 1930s, P. Du Val investigated these singularities and proved that the dual graph of their minimal resolution is exactly the Dynkin diagrams of type A_n, D_n, E_6, E_7, and E_8, corresponding to the cyclic groups, the binary dihedral groups, and the binary groups of motions of the tetrahedron, the octahedron, and the icosahedron. This was the first relation found between Kleinian singularities and the simple Lie algebras of type ADE. A natural question was whether this was a coincidence or there was a direct relation between them. Years later, in the 1960s, Brieskorn proved the existence of simultaneous resolutions for Kleinian singularities. After reading Brieskorn's work, Grothendieck conjectured that Kleinian singularities can be obtained from the corresponding simple Lie algebra of type A, D, or E, intersecting its nilpotent variety with a slice transverse to the orbit of a subregular element. The proof of Grothendieck's conjecture was announced by Brieskorn at the ICM in Nice 1970, with a sketch of the proof. In 1976, H. Esnault gave in her PhD thesis a complete proof of this theorem, following Grothendieck's initial ideas. Chapter 10 by José Luis Cisneros Molina and Meral Tosun discusses Brieskorn's theorem and a generalization of this for simple elliptic singularities which are non-hypersurface complete intersections. The chapter gives all the ingredients one needs to understand this beautiful piece of work. It discusses also several more recent developments and related topics, as the McKay correspondence, which describes how to obtain the Dynkin diagrams of type ADE from the irreducible representations of the corresponding finite subgroups of SU(2), giving a one-to-one correspondence between the nontrivial irreducible representations of the group and the components of the exceptional set of the minimal resolution of the associated Kleinian singularity.

So we see that the individual chapters cover a wide range of topics in singularity theory, and at the same time, they are linked to each other in fundamental ways.

Cuernavaca, Mexico — José Luis Cisneros Molina
Marseille, France — Dũng Tráng Lê
Mexico City, Mexico — José Seade
March 2020

Contents