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Analytic Number Theory

Lectures given at the
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Preface

The origins of analytic number theory, i.e. of the study of arithmetical problems by analytic methods, can be traced back to Euler's 1737 proof of the divergence of the series $\sum 1/p$ where p runs through all prime numbers, a simple, yet powerful, combination of arithmetic and analysis. One century later, during the years 1837-40, Dirichlet produced a major development in prime number theory by extending Euler's result to primes p in an arithmetic progression, $p \equiv a \pmod q$ for any coprime integers a and q. To this end Dirichlet introduced group characters χ and L-functions, and obtained a key result, the non-vanishing of $L(1, \chi)$, through his celebrated formula on the number of equivalence classes of binary quadratic forms with a given discriminant.

The study of the distribution of prime numbers was deeply transformed in 1859 by the appearance of the famous nine pages long paper by Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse*, where the author introduced the revolutionary ideas of studying the zeta-function $\zeta(s) = \sum_1^\infty n^{-s}$ (and hence, implicitly, also the Dirichlet L-functions) as an analytic function of the complex variable s satisfying a suitable functional equation, and of relating the distribution of prime numbers with the distribution of zeros of $\zeta(s)$. Riemann considered it highly probable ("sehr wahrscheinlich") that the complex zeros of $\zeta(s)$ all have real part $\frac{1}{2}$. This still unproved statement is the celebrated Riemann Hypothesis, and the analogue for all Dirichlet L-functions is known as the Grand Riemann Hypothesis. Several crucial results were obtained in the following decades along the way opened by Riemann, in particular the Prime Number Theorem which had been conjectured by Legendre and Gauss and was proved in 1896 by Hadamard and de la Vallée Poussin independently.

During the twentieth century, research subjects and technical tools of analytic number theory had an astonishing evolution. Besides complex function theory and Fourier analysis, which are indispensable instruments in prime number theory since Riemann's 1859 paper, among the main tools and contributions to analytic number theory developed in the course of last century one should mention at least the circle method introduced by Hardy, Littlewood and Ramanujan in the 1920's, and later improved by Vinogradov and by Kloosterman, as an analytic technique for the study of diophantine equations and of additive problems over primes or over special integer sequences, the sieve methods of Brun and Selberg, subsequently developed by Bombieri, Iwaniec and others, the large sieve introduced by Linnik and substantially modified and improved by Bombieri, the estimations of exponential sums due to Weyl, van der Corput and Vinogradov, and the theory of modular forms and automorphic L-functions.

The great vitality of the current research in all these areas suggested our proposal for a C.I.M.E. session on analytic number theory, which was held at Cetraro (Cosenza, Italy) from July 11 to July 18, 2002. The session consisted of four six-hours courses given by Professors J. B. Friedlander (Toronto), D. R. Heath-Brown (Oxford), H. Iwaniec (Rutgers) and J. Kaczorowski (Poznań). The lectures were attended by fifty-nine participants from several countries, both graduate students and senior mathematicians. The expanded lecture notes of the four courses are presented in this volume.

The main aim of Friedlander's notes is to introduce the reader to the recent developments of sieve theory leading to prime-producing sieves. The first part of the paper contains an account of the classical sieve methods of Brun, Selberg, Bombieri and Iwaniec. The second part deals with the outstanding recent achievements of sieve theory, leading to an asymptotic formula for the number of primes in certain thin sequences, such as the values of two-variables polynomials of type $x^2 + y^4$ or $x^3 + 2y^3$. In particular, the author gives an overview of the proof of the asymptotic formula for the number of primes represented by the polynomial $x^2 + y^4$. Such an overview clearly shows the role of bilinear forms, a new basic ingredient in such sieves.

Heath-Brown's lectures deal with integer solutions to Diophantine equations of type $F(x_1, \ldots, x_n) = 0$ with absolutely irreducible polynomials $F \in \mathbb{Z}[x_1, \ldots, x_n]$. The main goal here is to count such solutions, and in particular to find bounds for the number of solutions in large regions of type $|x_i| \leqslant B$. The paper begins with several classical examples, with the relevant problems for curves, surfaces and higher dimensional varieties, and with a survey of many results and conjectures. The bulk of the paper deals with the proofs of the main theorems where several tools are employed, including results from algebraic geometry and from the geometry of numbers. In the final part, applications to power-free values of polynomials and to sums of powers are given.

The main focus of Iwaniec's paper is on the exceptional Dirichlet character. It is well known that exceptional characters and exceptional zeros play a relevant role in various applications of the L-functions. The paper begins with a survey of the classical material, presenting several applications to the class number problem and to the distribution of primes. Recent results are then outlined, dealing also with complex zeros on the critical line and with families of L-functions. The last section deals with Linnik's celebrated theorem on the least prime in an arithmetic progression, which uses many properties of the exceptional zero. However, here the point of view is rather different from Linnik's original approach. In fact, a new proof of Linnik's result based on sieve methods is given, with only a moderate use of L-functions.

Kaczorowski's lectures present a survey of the axiomatic class S of L-functions introduced by Selberg. Essentially, the main aim of the Selberg class theory is to prove that such an axiomatic class coincides with the class of automorphic L-functions. Although the theory is rich in interesting conjectures, the focus of these lecture notes is mainly on unconditional results. After a chapter on classical examples of L-functions and one on the basic theory, the notes present an account of the invariant theory for S. The core of the theory begins with chapter 4, where the necessary material on hypergeometric functions is collected. Such results are applied in the following chapters, thus obtaining information on the linear and non-linear twists which, in turn, yield a complete characterization of the degree 1 functions and the non-existence of functions with degree between 1 and 5/3.

We are pleased to express our warmest thanks to the authors for accepting our invitation to the C.I.M.E. session, and for agreeing to write the fine papers collected in this volume.

Alberto Perelli Carlo Viola

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