J. P. Myers, Jr. M. J. O'Donnell (Eds.)

Constructivity in Computer Science

Summer Symposium
San Antonio, TX, June 19-22, 1991
Proceedings

Springer-Verlag
Berlin Heidelberg New York
London Paris Tokyo
Hong Kong Barcelona
Budapest

Series Editors

Gerhard Goos
Universität Karlsruhe
Postfach 69 80
Vincenz-Priessnitz-Straße 1
W-7500 Karlsruhe, FRG

Juris Hartmanis
Department of Computer Science
Cornell University
5149 Upson Hall
Ithaca, NY 14853, USA

Volume Editors

J. Paul Myers, Jr.
Department of Computer Science, Trinity University
715 Stadium Drive, San Antonio, TX 78212, USA

Michael J. O'Donnell
Department of Computer Science, The University of Chicago
Ryerson Hall, 1100 E. 58th Street, Chicago, IL 60637-2745, USA

CR Subject Classification (1991): F.4.1, I.2.3, F.3.2-3, D.3.1

ISBN 3-540-55631-1 Springer-Verlag Berlin Heidelberg New York
ISBN 0-387-55631-1 Springer-Verlag New York Berlin Heidelberg

Printed in Germany

Typesetting: Camera ready by author/editor
Printing and binding: Druckhaus Beltz, Hemsbach/Bergstr.
45/3140-543210 - Printed on acid-free paper

Preface

Mathematicians have long recognized the distinction between an argument showing that an interesting object exists and a procedure for actually constructing the object. Some reject nonconstructive proof of existence as invalid, but even those who accept nonconstructive proof usually value the additional insight given by a mental construction. Computer science adds a new dimension of interest in constructivity, since a computer program is a formal description of a constructive procedure that can be executed automatically. So, computer science motivates an interest in constructions as objects with useful behaviors, in addition to the mathematical interest in constructions as direct sources of insight. That constructivity has assumed much importance in computer science is reflected in the title of this symposium, mirroring the name of the first colloquium: "Constructivity in Mathematics" (Heyting, 1957).

The Symposium on Constructivity in Computer Science was sponsored by Trinity University, the University of Chicago, and the Association for Symbolic Logic. The symposium drew participation from Canada, France, Germany, the People's Republic of China, Sweden, the United Kingdom, and the United States of America. Topics discussed were quite diverse, including semantics and type theory, theorem proving, logic, analysis, topology, combinatorics, nonconstructive methods in graph theory, and a special track on curriculum and pedagogy.

This volume contains papers presented at the symposium. Preliminary written versions of papers were distributed, in addition to the lectures. The presentations stimulated very lively discussion, and the papers have been revised based on the feedback from those discussions.

Serge Yoccoz presented a paper, *Some Properties and Applications of the Lawson Topology*, which was not available for these proceedings. The paper of Thierry Coquand, who was unable to attend, was delivered by Chetan Murthy.

Acknowledgments

The Midstates Science and Mathematics Consortium of the Pew Charitable Trusts supported activities that led to the planning for the symposium through a 1989 faculty development award to Professor Myers.

Trinity University provided some funding and visa support for participants. And the University's William Knox Holt Continuing Education and Conference Center provided meeting and residential facilities as well as registration support.

The Association for Symbolic Logic, the American Mathematical Society, and the Special Interest Group on Automata and Computability Theory of the Association for Computing Machinery supported the symposium through free announcements in their periodicals.

And we appreciate encouragement and logistical support (e.g., scheduling assistance and mailing lists) from Robert Constable, Herbert Enderton, Ward Henson, Daniel Leivant, Albert Meyer, Yiannis Moschovakis, and Dana Scott.

May 1992

J. Paul Myers, Jr.
Michael J. O'Donnell

Participants

Jawahar Chirimar	University of Pennsylvania
Ching-Tsun Chou	UCLA
Thierry Coquand	Chalmers Tekniska Högskola
John Doner	University of California at Santa Barbara
Michael R. Fellows	University of Victoria
Newcomb Greenleaf	Columbia University
William Halchin	Software Engineering Consultant
Jim Hoover	University of Alberta
Douglas J. Howe	Cornell University
Jiafu Xu	Nanjing University
Jianguo Lu	Nanjing University
Stuart A. Kurtz	The University of Chicago
Michael A. Langston	University of Tennessee
James Lipton	University of Pennsylvania
Andrew Malton	Queen's University
John C. Mitchell	Stanford University
Chetan R. Murthy	Cornell University
J. Paul Myers, Jr.	Trinity University
Maria Napierala	Oregon Graduate Institute
Michael J. O'Donnell	The University of Chicago
Walter Potter	Southwestern University
Ronald E. Prather	Trinity University
Uday S. Reddy	University of Illinois at Urbana-Champaign
Scott F. Smith	The Johns Hopkins University
Ned Staples	Southeastern Massachusetts University
Yong Sun	University of York
Vipin Swarup	The MITRE Corporation
Simon Thompson	University of Kent at Canterbury
J.V. Tucker	University College of Swansea
Klaus Weihrauch	Fern Universität
Judith Wilson	Temple University
Serge Yoccoz	LABRI, Université de Bordeaux
J.I. Zucker	McMaster University

Contents

Semantics

Types

Logic

Nonconstructive Techniques

Topology

Reals

Curriculum / Pedagogy

Connecting Formal Semantics to Constructive Intuitions

Stuart A. Kurtz
The University of Chicago

John C. Mitchell
Stanford University

Michael J. O'Donnell
The University of Chicago

1 Abstract

We use formal semantic analysis to generate intuitive confidence that the Heyting Calculus is an appropriate system of deduction for constructive reasoning. Well-known modal semantic formalisms have been defined by Kripke and Beth, but these have no formal concepts corresponding to constructions, and shed little intuitive light on the meanings of formulae. In particular, the well-known completeness proofs for these semantics do not generate confidence in the sufficiency of the Heyting Calculus, since we have no reason to believe that every intuitively constructive truth is valid in the formal semantics.

Läuchli has proved completeness for a realizability semantics with formal concepts analogous to constructions, but the analogy is inherently inexact. We argue that, in spite of this inexactness, every intuitively constructive truth is valid in Läuchli semantics, and therefore the Heyting Calculus is powerful enough to prove all constructive truths. Our argument is based on the postulate that a uniformly constructible object must be communicable in spite of imprecision in our language, and we show how the permutations in Läuchli's semantics represent conceivable imprecision in a language, independently of the particular structure of the language.

We look at some of the details of a generalization of Läuchli's proof of completeness for the propositional part of the Heyting Calculus, in order to expose the required model constructions and the constructive content of the result. We discuss the reasons why Läuchli's completeness results on the predicate calculus are not constructive.

2 General Introduction

This paper presents a detailed outline of a three-part lecture given by Michael J. O'Donnell at the symposium on Constructivity in Computer Science. The lecture describes collaborative work in progress by the three authors above, attempting to use formal proofs of completeness for the Heyting Calculus to provide *intuitive* confidence that all constructively true propositional formulae are provable. Feedback from the symposium participants, and particularly a very detailed and cogent critique from James Lipton of the University of Pennsylvania, helped substantially in improving the presentation and the scholarship of the work.

The speaker tried to stimulate thinking on a number of side topics, and to connect to many of the issues raised in other papers at the conference, but did not attempt a thorough survey of the area. Some technical improvements to formal systems and proofs of completeness are introduced, but they are all variations of previously known results. The goal of the lecture is thorough understanding of known technical results and their connection to intuitive concepts, rather than new technicalities, or a thorough survey of known results.

Section 3 presents the lecture as an outline, rather than a narrative. Part I introduces the basic intuitions of constructivism, and describes the types of insights that we hope to get from a formal semantic treatment. It describes Kripke's and Beth's formal semantics for constructive logic, and explains why these do *not* give the desired insights. Part II introduces the realizability and formulae-as-types approaches to semantics. It defines Läuchli's version of realizability, based on permutation-invariant functions, and explains permutation invariance as a plausible ***necessary condition*** for reliable communicability, and therefore for constructibility, of a function. Finally, Part III gives the formal proof that the Heyting Propositional Calculus is complete for Läuchli's realizability semantics.

3 The Lecture

I. First part: Introduction to constructivism, the intuitive use of semantics.

I.A. What is "constructivism"?

I.A.1. We seek a useful formalism for a reasonable constructive philosophy, not a treatment of a particular historical school, such as intuitionism.

——**2.** The basic intuition of constructivism is that, by asserting the proposition α we claim to have a mental construction verifying α. The precise meaning of "construction" is problematic. A number of philosophers and mathematicians have discussed the problem, including Heyting [19], Dummett [7], Beeson [1], Kleene [25].

——**3.** Here are some examples of classically true formulae that are rejected by constructive logic. They are not all equivalent. See [22, 7, 50, 47] for discussion of their various strengths.

I.A.3.a. $\alpha \vee \neg\alpha$ *(excluded middle)*

———**b.** $\alpha \vee (\alpha \Rightarrow \beta)$

———**c.** $\neg\neg\alpha \Rightarrow \alpha$ *(double negation elimination)*

———**d.** $((\alpha \Rightarrow \beta) \Rightarrow \alpha) \Rightarrow \alpha$ *(Peirce's law [36])*

———**e.** $\neg\alpha \vee \neg\neg\alpha$

———**f.** $(\alpha \Rightarrow \beta) \vee ((\alpha \Rightarrow \beta) \Rightarrow \alpha)$

———**g.** $(\alpha \Rightarrow \beta) \vee (\beta \Rightarrow \alpha)$

———**h.** $((\alpha \Rightarrow \beta) \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \alpha) \Rightarrow \gamma) \Rightarrow \gamma$

———**i.** $\alpha \vee (\alpha \Rightarrow \beta) \vee \neg\beta$

———**j.** $\neg\alpha \vee \neg\neg\alpha \vee (\alpha \Rightarrow (\neg\beta \vee \neg\neg\beta))$

I.A.4. Formal system: Heyting Propositional Calculus [18]—essentially Classical Propositional Calculus without the law of excluded middle.

I.B. Semantics: the study of meaning.

I.B.1. The use of semantics: justify and explain a formal system of inference by clarifying its connection to ***intuitive*** meaning. The intensional structure of formal semantics, not the mere extension of the class of true formulae, connects formalism to intuitive meaning[1]. We must inspect carefully and rigorously, but informally, the connection between formal semantics and intuitive meaning, then examine formally the connection between formal semantics and a formal system of inference.

I.B.2. Notation: Given a fixed system of proof, let F be a formal semantic system. F provides a set of possible interpretations for atomic propositional symbols and, for each interpretation, criteria for marking certain formulae "true". Let α be a formula, and let Γ be a set of formulae.

- $\Gamma \vdash \alpha$ means that α is provable when Γ are assumed.
- $\Gamma \models \alpha$ means that α holds *intuitively* whenever the assumptions in Γ hold.
- $\Gamma \models_F \alpha$ means that α is marked true in every F-interpretation for which all assumptions in Γ are marked true. $\models_F$ is the ***logical consequence*** relation induced by F.

I.B.3. Intuitive vs. formal measures of strength of a formal system.

I.B.3.a. Faithfulness[2]: $\Gamma \vdash \alpha$ implies $\Gamma \models \alpha$.

———**b.** Soundness: $\Gamma \vdash \alpha$ implies $\Gamma \models_F \alpha$.

———**c.** Fullness: $\Gamma \models \alpha$ implies $\Gamma \vdash \alpha$.

———**d.** Completeness: $\Gamma \models_F \alpha$ implies $\Gamma \vdash \alpha$.

I.B.4. To go from soundness for $\models_F$ to faithfulness for $\models$, we need to show intuitively but rigorously that $\Gamma \models_F \alpha$ implies $\Gamma \models \alpha$. That is, $\models_F$ is a *lower bound* for $\models$. This argument is usually very simple.

——**5.** To go from completeness for $\models_F$ to fullness for $\models$, we need to show that $\Gamma \models \alpha$ implies $\Gamma \models_F \alpha$. That is, $\models_F$ is an ***upper bound*** for $\models$. This argument is usually difficult, and may

[1] Tarski's classical semantics [44] are useful largely because they reveal meaning from syntactic structure, although Tarski claimed that he only wanted to mark the true formulae.

[2] Feferman has related concepts of faithfulness and adequacy [9, 1].