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The Logic of Information Structures

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Preface

The present monograph is a revised version of my doctoral thesis at the Fachbereich Philosophie und Sozialwissenschaften I of the Free University of Berlin. It contains my contribution to the interdisciplinary research project "Systeme der Logik als theoretische Grundlage der Wissens- und Informationsverarbeitung" at the Institute for Philosophy of the Free University of Berlin.

I am very glad that this preface gives me the opportunity to express my gratitude to various people and institutions. First of all, I would like to thank my thesis advisers, David Pearce and Johan van Benthem. With each of them I associate one of the basic ideas which in combination led me to writing this book, viz. (i) to regard negative and positive information as equally relevant and (ii) to vary in a systematic way structural rules of inference. Without their inspiration this thesis would not have come into existence, and it would not have been completed without Johan van Benthem's critical comments and encouragement. Moreover, I gratefully acknowledge scholarships from the Senat von Berlin and the Studienstiftung des deutschen Volkes. To the latter I am indebted for support over many years. A number of colleagues have, in one way or another, contributed to the realization of this book. I would like to thank Kosta Došen and Dirk Roorda for their critical remarks on an early version of Chapter 5 and several stimulating discussions. To Dirk in addition I return thanks for practical help concerning 'huisvesting'. I would also like to thank Peter Schroeder-Heister. His insistance on elimination rules enormously helped me to understand the proof-theoretic approach towards the problem of functional completeness. André Fuhrmann was so kind to comment on an early version of parts of Chapter 9. I also wish to thank Gerd Wagner for numerous conversations and three referees of the *Journal of Logic, Language and Information* for their reports. Finally, I am grateful to Jörg Siekmann for recommending this book for Springer Lecture Notes in AI. From the very beginning I could not have carried out this work without the support from Petra, Kasimir and my parents. They deserve my special, whole-hearted thanks.

As indicated in the text, Chapter 5 is based on 'Formulas-as-types for a hierarchy of sublogics of intuitionistic propositional logic' in D. Pearce & H. Wansing (eds.), *Nonclassical Logics and Information Processing*, Lecture Notes in AI, Vol. 619, Springer-Verlag, Berlin, 1992, and Chapter 4 (the part consisting of Sections 4.1 to 4.4) is (based on) my paper 'Functional completeness for subsystems of intuitionistic propositional logic', forthcoming in the *Journal of Philosophical Logic*. Chapter 4, in particular Section 4.5, is also used in 'On the expressiveness of Categorial Grammar', to appear in J. Wolenski (ed.), *The Legacy of Ajdukiewicz*, Rodopi, Amsterdam. Material from, essentially, Chapter 9 will appear as 'Informational Interpretation of Substructural Propositional Logics' in the *Journal of Logic, Language and Information.*

Hamburg, March 1993 Heinrich Wansing

Contents

Chapter 1

Introduction

The concept of information can be studied from numerous points of view. Without doubt, however, *information structure* and *information processing* form central aspects of the study of information. Whereas information structure can be regarded as a subject of *model theory*, information processing may be viewed as a matter of *proof theory*. The present investigation pursues this logical perspective. It can be considered a systematic contribution to the line of research that began with S. Kripke's [1965] interpretation of intuitionistic logic in models based on pre-ordered information states. The following table identifies the most important topics that will be dealt with.

proof theory	the systematic variation of structural inference rules in sequent calculi, which offers various options for representing premises as databases and the sequent arrow $\to$ as an information-processing mechanism (Chapters 2 and 3)
	cut-elimination and consequences thereof (Chapters 3 and 6)
	functional completeness wrt a proof-theoretic interpretation of logical operations (Chapters 4 and 7)
model theory	the encoding of proofs by typed λ-terms and vice versa (Chapters 5 and 8)
	information models, i.e. models based on certain abstract information structures, where by an abstract information structure we understand a non-empty set I viewed as a set of *information pieces* or *information states* represented by pieces of information together with certain relations or operations on I, possibly some designated pieces of information, and possibly certain conditions on these relations, operations or designated elements (Chapters 1, 2 and 9)

Table 1.1: The main topics.

A general theme, which will be alluded to in considerations on information processing as well as information structure, is the dichotomy between *positive* and *negative* information (Chapters 2 and 6 - 9). The central claims are that both positive and negative information should be treated in their own right as independent and equally relevant concepts, and that this position leads to strong, *constructive* negation. The first two chapters prepare the stage for a uniform and more comprehensive discussion of information structure and deductive information processing in the remaining chapters by providing examples and motivation.

The whole investigation is concerned with propositional logics only. The propositional sequent calculi considered can easily be extended to predicate logics by adding the usual rules for the existential and universal quantifiers (and, in the presence of strong, constructive negation, the usual rules for their strongly negated forms). We will make use of $\exists$ and $\forall$ as quantifiers in the metalanguage. The metalogic used is classical; repeatedly there will be applications of classical reductio ad absurdum as a rule. Where misunderstandings are unlikely to arise, sometimes no special attention will be paid to the distinction between the mention and use of symbols.

1.1 Intuitionistic propositional logic IPL

An obvious starting point for investigating logics of information structures is reviewing their most famous exponent, viz. intuitionistic propositional logic IPL. Preparatory to the introduction of various formal systems in later chapters, we shall first give a presentation of IPL in perhaps somewhat unorthodox language.

Definition 1.1 The vocabulary of the propositional language L consists of

a denumerable set $PROP$ of propositional variables;
two verum constants: $\mathbf{t}$, $\top$;
one falsum constant: $\bot$;
binary connectives: / (right-searching implication), \ (left-searching implication),
$\circ$ (intensional conjunction), $\wedge$ (extensional conjunction), $\vee$ (disjunction);
auxiliary symbols: (,).

Definition 1.2 The set of L-formulas is the smallest set Γ such that

$PROP \subseteq \Gamma$;

$\mathbf{t}, \top, \bot \in \Gamma$;

if $A, B \in \Gamma$, then $(A/B), (A \setminus B), (A \circ B), (A \wedge B), (A \vee B) \in \Gamma$.

We use $p, p_1, p_2, \ldots$ etc. to denote propositional variables, A, B, C, A_1, A_2, ... etc. to denote L-formulas, and X, Y, Z, X_1, X_2, ... etc. to denote finite, possibly empty sequences of L-formula occurrences. Sometimes $<>$ will be used to denote the empty sequence. Outermost parentheses of formulas will not always be written.

Definition 1.3 The notion of a subformula of A is inductively defined as follows:

every L-formula A is a subformula of itself;

the subformulas of A and the subformulas of B are subformulas of (B/A), $(A \setminus B)$, $(A \circ B)$, $(A \wedge B)$, and $(A \vee B)$.

An expression $X \rightarrow A$ is called a sequent; X is called its antecedent and A its succedent. In case that $n = 0$, $A_1 \ldots A_n \rightarrow A$ denotes $\rightarrow A$. Negation in IPL is a defined notion, we have $\neg^r A \stackrel{def}{=} (\bot/A)$, $\neg^l A \stackrel{def}{=} (A \setminus \bot)$. $A \rightleftharpoons^+ B$ is used as an abbreviation for $(A \setminus B) \wedge (B \setminus A) \wedge (B/A) \wedge (A/B)$. Next, we present IPL as a symmetric sequent calculus with (i) logical rules, (ii) operational rules for introducing connectives on the left hand side (lhs) and on the right hand side (rhs) of the sequent arrow $\rightarrow$, and (iii) a number of structural inference rules.

Definition 1.4 The rules constituting IPL are:

logical rules :	
(id)	$\vdash A \rightarrow A,$
(cut)	$Y \rightarrow A \quad XAZ \rightarrow B \vdash XYZ \rightarrow B,$
operational rules :	
$(\bot \rightarrow)$	$\vdash X \bot Y \rightarrow A,$
$(\rightarrow \mathbf{t})$	$\vdash X \rightarrow \mathbf{t},$
$(\rightarrow \top)$	$\vdash \rightarrow \top,$
$(\top \rightarrow)$	$XY \rightarrow A \vdash X \top Y \rightarrow A,$
$(\rightarrow /)$	$XA \rightarrow B \vdash X \rightarrow (B/A),$
$(/ \rightarrow)$	$Y \rightarrow A \quad XBZ \rightarrow C \vdash X(B/A)YZ \rightarrow C,$
$(\rightarrow \setminus)$	$AX \rightarrow B \vdash X \rightarrow (A \setminus B),$
$(\setminus \rightarrow)$	$Y \rightarrow A \quad XBZ \rightarrow C \vdash XY(A \setminus B)Z \rightarrow C,$
$(\rightarrow \circ)$	$X \rightarrow A \quad Y \rightarrow B \vdash XY \rightarrow (A \circ B),$
$(\circ \rightarrow)$	$XABY \rightarrow C \vdash X(A \circ B)Y \rightarrow C,$
$(\rightarrow \wedge)$	$X \rightarrow A \quad X \rightarrow B \vdash X \rightarrow (A \wedge B),$
$(\wedge \rightarrow)$	$XAY \rightarrow C \vdash X(A \wedge B)Y \rightarrow C,$
	$XBY \rightarrow C \vdash X(A \wedge B)Y \rightarrow C,$
$(\rightarrow \vee)$	$X \rightarrow A \vdash X \rightarrow (A \vee B),$
	$X \rightarrow B \vdash X \rightarrow (A \vee B),$
$(\vee \rightarrow)$	$XAY \rightarrow C \quad XBY \rightarrow C \vdash X(A \vee B)Y \rightarrow C;$
structural rules :	
permutation $(\mathbf{P})$:	$XABY \rightarrow C \vdash XBAY \rightarrow C;$
contraction $(\mathbf{C})$:	$XAAY \rightarrow B \vdash XAY \rightarrow B;$
monotonicity $(\mathbf{M})$:	$XY \rightarrow B \vdash XAY \rightarrow B.$

The rules $(\to /)$ and $(\to \backslash)$ are directional versions of the deduction theorem. The notion of a derivation in IPL of $X \to A$ from a finite, possibly empty sequence of sequent occurrences is defined by induction on the rules of IPL, see Appendix 1.5. If Π is a derivation in IPL of $X \to A$ from the empty sequence, then Π is called a proof of $X \to A$ in IPL, and if there is a proof of $X \to A$ in IPL, this is denoted by $\vdash_{IPL} X \to A$. If the context is clear, we shall sometimes just write $\vdash X \to A$. A proof of $\to A$ in IPL is also called a proof of A in IPL. If there is a proof of A in IPL, then A is called a theorem of IPL. Two formulas A and B are said to be interderivable in IPL, if $\vdash A \to B$ and $\vdash B \to A$, which is abbreviated by $\vdash A \leftrightarrow B$. One can easily show that $\vdash \to A \rightleftharpoons^{+} B$ iff $\vdash A \leftrightarrow B$.

Since **P** is present, directional implications (A/B) and $(B \setminus A)$ resp. directional negations $\neg^{r} A$ and $\neg^{l} A$ are interderivable in IPL. Due to the presence of **M**, the verum constants **t** and $\top$ are interderivable. Moreover, since **C** and **M** are available, also $(A \circ B)$ and $(A \wedge B)$ are interderivable. Thus, in the presence of the structural rules **P**, **C**, and **M** one could do without **t** and intensional conjunction $\circ$, and one could replace the two directional implications $/, \backslash$ resp. negations $\neg^{r}$, $\neg^{l}$ by the more usual implication sign $\supset$ resp. $\neg$.[1] Note that $\top$ (and hence **t**) is definable in IPL as (p/p), for some propositional variable p. In the sequel we shall sometimes use $\supset$ instead of $/, \backslash$ and $\neg$ instead of $\neg^{r}$, $\neg^{l}$, and forget about $\circ$, **t**, and $\top$, if, like in IPL, **P**, **C**, and **M** are assumed to be available. Note also that since **P**, **C**, and the structural inference rule

$$\text{expansion } (\mathbf{E}): \quad XAY \to B \vdash XAAY \to B$$

as a special case of **M** are present, the sequences on the lhs of $\to$ may be conceived of as finite *sets*.

EXAMPLE As an example of a derivation in IPL we prove the distribution of $\wedge$ over $\vee$, i.e. $A \wedge (B \vee C) \to (A \wedge B) \vee (A \wedge C)$, using **M** and **C**:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{A \to A}{AB \to A} \quad \dfrac{B \to B}{AB \to B}}{AB \to A \wedge B}}{AB \to (A \wedge B) \vee (A \wedge C)} \quad \dfrac{\dfrac{\dfrac{A \to A}{AC \to A} \quad \dfrac{C \to C}{AC \to C}}{AC \to A \wedge C}}{AC \to (A \wedge B) \vee (A \wedge C)}}{A(B \vee C) \to (A \wedge B) \vee (A \wedge C)}}{(A \wedge (B \vee C))(B \vee C) \to (A \wedge B) \vee (A \wedge C)}}{(A \wedge (B \vee C))(A \wedge (B \vee C)) \to (A \wedge B) \vee (A \wedge C)}}{(A \wedge (B \vee C)) \to (A \wedge B) \vee (A \wedge C).}$$

1.2 Kripke's interpretation of IPL

We shall briefly describe Kripke's semantics for IPL and reproduce its interpretation in terms of information states as suggested by Kripke.

Definition 1.5 A Kripke frame is a structure $\mathcal{F} = < I, \sqsubseteq >$, where I is a non-empty set and $\sqsubseteq$ is a pre-order (or quasi-order) on I, i.e. $\sqsubseteq$ is a reflexive and transitive binary relation on I.

[1] This is justified on the strength of a replacement theorem that will be proved in Chapter 3.

Definition 1.6 A minimal Kripke model based on a Kripke frame $\mathcal{F}$ is a structure $\mathcal{M} = <\mathcal{F}, v_0>$, where v_0 is a basic valuation function from $PROP \cup \{\bot\}$ into 2^I such that for every $p \in PROP \cup \{\bot\}$ and every $a, b \in I$:

$$\text{if } a \sqsubseteq b, \text{ then } a \in v_0(p) \text{ implies } b \in v_0(p).$$

Definition 1.7 An intuitionistic Kripke model based on a Kripke frame $\mathcal{F}$ is a minimal Kripke model $\mathcal{M} = <\mathcal{F}, v_0>$, where $v_0(\bot) = \emptyset$.

Definition 1.8 Given a Kripke model (minimal or intuitionistic) $\mathcal{M} = <I, \sqsubseteq, v_0>$, v_0 is inductively extended to a valuation function v from the set of all L-formulas into 2^I as follows:

$$\begin{array}{lcllll}
v(p) & = & v_0(p), & & p \in PROP \cup \{\bot\} \\
v(A \wedge B) & = & v(A \circ B) & = & v(A) \cap v(B), \\
v(A \vee B) & = & v(A) \cup v(B), & & \\
v(A \setminus B) & = & v(B/A) & = & \{a \in I \mid (\forall b \in v(A))\, a \sqsubseteq b \text{ implies } b \in v(B)\}, \\
v(\top) & = & v(\mathbf{t}) & = & I.
\end{array}$$

By induction on the complexity of A it can easily be shown that for every Kripke model $\mathcal{M} = <I, \sqsubseteq, v_0>$, every L-formula A, and every a, $b \in I$:

(Heredity) if $a \sqsubseteq b$, then $a \in v(A)$ implies $b \in v(A)$.

Definition 1.9 (semantic consequence) Let $\mathcal{M} = <I, \sqsubseteq, v_0>$ be a Kripke model. A sequent $s = A_1 \dots A_n \to A$ holds (or is valid) at $a \in I$

$$\text{iff } \begin{cases} a \in v(A_1 \circ \dots \circ A_n) \text{ implies } a \in v(A) & \text{if } n > 0 \\ a \in v(A) & \text{otherwise.} \end{cases}$$

The sequent s holds (or is valid) in $\mathcal{M}$ iff s holds at every $a \in I$. If $\to A$ holds at $a \in \mathcal{M}$ resp. is valid in $\mathcal{M}$, then also A is said to hold at $a \in \mathcal{M}$ resp. to be valid in $\mathcal{M}$. The sequent s holds (or is valid) in IPL iff s holds in every intuitionistic Kripke model. If $\to A$ is valid in IPL, then also A is said to be valid in IPL.

If $A_1 \dots A_n \to A$ is valid in IPL, then for every intuitionistic Kripke model $\mathcal{M} = <I, \sqsubseteq, v_0>$, $v(A_1 \circ \dots \circ A_n) \subseteq v(A)$. This notion of validity may be contrasted with the weaker requirement that if $A_1, \dots, A_n$ are valid in IPL, then A is valid in IPL.

The elements of I can, according to Kripke, be thought of as "points in time (or 'evidential situations'), at which we may have various pieces of information" [1965, p. 98]. We may also identify a state $a \in I$ with the pieces of information available at a. A propositional variable p is verfied at $a \in I$, i.e. $a \in v(p)$, iff there is enough information at a to prove p. Thus, $a \notin v_0(p)$ does not mean that p is falsified at a, it merely says p is not verifed at a. The verification of complex L-formulas at $a \in I$ is determined by the definition of v, given a basic valuation v_0. Since $\neg^r A$ resp. $\neg^l A$ is defined as $\bot/A$ resp. $A \setminus \bot$, for intuitionistic Kripke models we have:

$$\begin{aligned} v(\neg^r A) &= v(\neg^l A) \\ &= \{a \in I \mid (\forall b \in v(A)) \text{ if } a \sqsubseteq b, \text{ then } b \in \emptyset\} \\ &= \{a \in I \mid (\forall b \in I) \text{ if } a \sqsubseteq b, \text{ then } b \notin v(A)\}. \end{aligned}$$

As we are dealing with evidential situations in pre-ordered time, these situations or information states may develop differently depending on the basic information aquired in the course of time. Thus, $a \sqsubseteq b$ says that information state a may develop into information state b, and transitivity of $\sqsubseteq$ becomes rather obvious, intuitively. Moreover, it is assumed that every $b \in I$ may develop into itself, since the information available at b "may be all the knowledge we have for an arbitrarily long time" [Kripke 1965, p. 99]. Eventually, because of (Heredity), information is never lost during the journey through time. Thus, 'possible development' is to be understood as 'possible expansion'.

Theorem 1.10 *IPL* is characterized by the class of all intuitionistic Kripke models, i.e. $\vdash_{IPL} A_1 \ldots A_n \rightarrow A$ iff $A_1 \ldots A_n \rightarrow A$ is valid in *IPL*.

Soundness, i.e. the 'only if' direction, can be proved by induction on the length of proofs in *IPL*. (Note that the rule M is validity-preserving because $\circ$, which is used to define the evaluation of sequents in Kripke models, is evaluated in exactly the same way as $\wedge$.) Using semantic tableaux, Kripke [1965] shows that every theorem of *IPL* is valid in every intuitionistic Kripke model. We shall sketch a proof of the completeness part of the above theorem (i.e. the 'if' direction) by defining a canonical intuitionistic Kripke model $\mathcal{M}_{IPL} = < I, \sqsubseteq, v_0 >$, i.e. a model which itself characterizes *IPL* (cf. e.g. [Tennant 1978, p. 106 ff.], [Došen 1989, p. 42 f.]). Let Γ be a set of L-formulas; Γ is deductively closed under *IPL* iff $\Gamma = \Gamma \cup \{A \mid \vdash_{IPL} A_1 \ldots A_n \rightarrow A \text{ and } A_i \in \Gamma\ (1 \leq i \leq n)\}$. Γ is said to be *IPL*-consistent iff for no sequence $A_1 \ldots A_n$, $A_i \in \Gamma$, $A_1 \ldots A_n \rightarrow \bot$ is provable in *IPL*. Γ is called prime iff for all L-formulas A, B: $(A \vee B) \in \Gamma$ implies $A \in \Gamma$ or $B \in \Gamma$. The canonical model $\mathcal{M}_{IPL}$ is defined as follows:

- $I = \{a \mid a \text{ is a prime, and } IPL\text{-consistent set of } L\text{-formulas deductively closed under } IPL\}$,
- $\sqsubseteq$ is the subset relation $\subseteq$,
- $v_0(p) = \{a \in I \mid p \in a\}$,
- $v_0(\bot) = \emptyset$.

Obviously, $\mathcal{M}_{IPL}$ is in fact an intuitionistic Kripke model. It can now be shown that if $\not\vdash_{IPL} A_1 \ldots A_n \rightarrow A$, then $A_1, \ldots, A_n$ belong to a prime, *IPL*-consistent set deductively closed under *IPL* which does not contain A. Using this fact, one can prove that for $\mathcal{M}_{IPL} = < I, \sqsubseteq, v_0 >$ the following holds for every L-formula A and every $a \in I$:

(Canon) $a \in v(A)$ iff $A \in a$.

By means of (Canon), completeness can easily be derived. If $A_1 \ldots A_n \rightarrow A$ is valid in every intuitionistic Kripke model, in particular it is valid in $\mathcal{M}_{IPL}$. Thus in $\mathcal{M}_{IPL}$, $v(A_1 \circ \ldots \circ A_n) \subseteq v(A)$, if $n > 0$, and $v(A) = I$, otherwise. By (Canon), $\vdash_{IPL} A_1 \circ \ldots \circ A_n \rightarrow A$ and thus $\vdash_{IPL} A_1 \ldots A_n \rightarrow A$, by $(\rightarrow \circ)$ and (*cut*).

In contrast to the situation in intuitionistic Kripke models, $\perp$ *may* hold at information pieces in minimal Kripke models. As a result of this interpretation of $\perp$, a sequent $X \perp Y \to A$ is not valid in every minimal Kripke model. The logic characterized by the class of all minimal Kripke models is Johansson's [1937] intuitionistic minimal propositional logic MPL.[2]

Definition 1.11 The rules of MPL are those of IPL without $(\perp \to)$.

In MPL nothing particular is assumed about $\perp$. The falsum constant $\perp$ can therefore be viewed just as a designated propositional variable used to define intuitionistic minimal negations $\neg^r$, $\neg^l$. The notions of derivation and proof in MPL are defined in the same way as for IPL.

Theorem 1.12 $\vdash_{MPL} A_1 \ldots A_n \to A$ iff $A_1 \ldots A_n \to A$ is valid in every minimal Kripke model.

This can be proved in strict analogy to the above proof for IPL. In the canonical model $\mathcal{M}_{MPL}$ for MPL, however, the pieces of information are prime sets of L-formulas deductively closed under MPL which need not be MPL-consistent, and the requirement that $v_0(\perp) = \emptyset$ is dropped.

In the construction of the canonical models $\mathcal{M}_{IPL}$ and $\mathcal{M}_{MPL}$ the defined relation $\sqsubseteq$ is not only a quasi-order but even a partial order on the set of information pieces, i.e. it is also anti-symmetric. Therefore IPL resp. MPL is also characterized by the class of all intuitionistic resp. minimal Kripke models based on a partially ordered set (*poset*). Moreover, there is a standard validity-preserving operation on Kripke models (see e.g. [Kripke 1965]) which applied to an intuitionistic resp. minimal Kripke model based on a *poset* produces an intuitionistic resp. minimal Kripke model $< I, \sqsubseteq, 1, v_0 >$ with $1 \in I$ and where $< I, \sqsubseteq, 1 >$ is a tree (i.e. $< I, \sqsubseteq >$ is a *poset* such that (i) 1 is an initial node: there is no $a \in I$ such that $a \neq 1$ and $a \sqsubseteq 1$, (ii) for every $a, b, c \in I$, if $a \sqsubseteq c$ and $b \sqsubseteq c$, then $a \sqsubseteq b$ or $b \sqsubseteq a$, and (iii) for each $a \in I$. $1 \sqsubseteq_n a$, where $\sqsubseteq_n$ is inductively defined as follows: for every $a, c \in I$, $a \sqsubseteq_0 c$ iff $a = c$; $a \sqsubseteq_1 c$ iff $a \sqsubseteq b$; $a \sqsubseteq_{n+2} b$ iff there is a $c \in I$ such that $a \sqsubseteq_{n+1} b$ and $b \sqsubseteq c$). Thus, IPL resp. MPL is also characterized by the class of all intuitionistic resp. minimal Kripke models based on a tree.[3] Kripke's interpretation can immediately be extended to Kripke models based on a tree: the initial node 1 is to be interpreted as the *initial* piece of information. In Kripke models based on a tree, by (Heredity), the evaluation clause for $\top$ can equivalently be formulated as:

$v(\top) = \{a \mid 1 \sqsubseteq a\}$,

and a sequent $A_1 \ldots A_n \to A$ can equivalently be said to be valid in a Kripke model $< I, \sqsubseteq, 1, v_0 >$

[2] The implication, negation fragment of MPL was first axiomatized by Kolmogorov [1925].

[3] There is also a standard validity preserving operation converting any Kripke model based on a quasi-ordered set into a Kripke model based on a *poset* (see [Kripke 1965]). Moreover, using a technique which is usually called 'unraveling', any Kripke model based on a tree can be converted into a Kripke model based on a finite tree validating exactly the same L-formulas (see 'selective filtration' in [Gabbay 1981, p. 69 f.]).

$$\text{iff} \begin{cases} 1 \in v(A_1 \circ \ldots \circ A_n) \text{ implies } 1 \in v(A) & \text{if } n > 0 \\ 1 \in v(A) & \text{otherwise.} \end{cases}$$

Thus, a sequent $\rightarrow A$ is provable in IPL resp. MPL iff $\top \rightarrow A$ is provble in IPL resp. MPL iff in every intuitionistic Kripke model resp. minimal Kripke model $< I, \sqsubseteq, 1, v_0 >$, A holds at 1.

The fact that IPL resp. MPL is characterized by the class of all intuitionistic resp. minimal Kripke models based on a tree can be used to show that IPL and MPL enjoy the following form of the disjunction property:

$\diamondsuit \quad \vdash X \rightarrow A \vee B$ iff $(\vdash X \rightarrow A$ or $\vdash X \rightarrow B)$

(see [van Dalen 1983, p. 186 f.]). We shall now give another proof of (strong) completeness of MPL wrt the class of all minimal Kripke models. For this purpose we will define the canonical model $\mathcal{M}'_{MPL}$ for MPL.

Definition 1.13 The canonical model $\mathcal{M}'_{MPL} =< I, \sqsubseteq, v_0 >$ is defined as follows:

- $I = \{a \mid \exists X = A_1 \ldots A_n (n \geq 0)$ and $a = \{A \mid \vdash_{MPL} X \rightarrow A\}\}$;
- $\sqsubseteq = \subseteq$;
- $v_0(p) = \{a \in I \mid p \in a\}$, for every p in $PROP \cup \{\perp\}$.

It can readily be verified that $\mathcal{M}'_{MPL}$ is in fact a minimal Kripke model. By induction on the complexity of A it can be shown that (Canon) holds for $\mathcal{M}'_{MPL}$. (We use the fact that MPL satisfies $\diamondsuit$.)

Theorem 1.14 $\vdash_{MPL} A_1 \ldots A_n \rightarrow A$ iff $A_1 \ldots A_n \rightarrow A$ is valid in every minimal Kripke model.

In order to prove completeness, assume that $A_1 \ldots A_n \rightarrow A$ is valid in every minimal Kripke model. Then $A_1 \ldots A_n \rightarrow A$ is valid in $\mathcal{M}'_{MPL}$. Thus, by (Canon), for every $a \in I$, $A_1 \circ \ldots \circ A_n \in a$, implies $A \in a$, if $n > 0$, and $A \in a$, otherwise. By the definition of I, this implies that for every sequence of L-formulas X, if $X \rightarrow A_1 \circ \ldots \circ A_n$ is provable in MPL, then $X \rightarrow A$ is provable in MPL. In particular $A_1 \circ \ldots \circ A_n \rightarrow A$ is provable in MPL.

Note that in $\mathcal{M}'_{MPL}$ every piece of information is finitely represented.

1.3 Grzegorczyk's interpretation of IPL

A less well-known semantics for IPL in terms of information pieces has been developed by Grzegorczyk. According to Grzegorkzyk [1964, p. 596] "intuitionistic logic can be understood as the logic of scientific research", where a "scientific research (e.g. an experimental investigation) consists of the successive enrichment of the set of data by new established facts obtained by means of our method of inquiry". In the retrospective, Grzegorczyk's approach to intuitionistic logic constitutes a concrete version of the characterization of IPL by intuitionistic Kripke models based on a tree. Grzegorczyk's approach is *concrete* in the sense that (i) it gives a concrete interpretation to the possible worlds or information pieces instead of taking them as primitive, (ii) for a particular set of information pieces it specifies a particular binary relation on them, and (iii) it specifies a basic valuation function $v_0 : PROP \cup \{\perp\} \longrightarrow 2^I$. In Grzegorczyk's case we have:

- every finite set of propositional variables is a possible world interpreted as a piece of information;
- let I be a nonempty set of information pieces, and let P be a mapping from I in nonempty subsets of I such that

  (*) if $a = \{p_1, \ldots, p_n\} \in I$, then either $P(a) = \{a\}$ or for every $b \in P(a)$ there exist $p_{n+1}, \ldots, p_{n+k+1}$ $(k \geq 0)$ such that $b = \{p_1, \ldots, p_n, p_{n+1}, \ldots, p_{n+1+k+1}\}$.

  P is interpreted as "the function of possible prolongations of the informations" in I. A binary relation $\sqsubseteq$ on I ("extension of information") is defined in terms of P as follows: for every a, $b \in I$,

  $a \sqsubseteq^0 b$ iff $a = b$;

  $a \sqsubseteq^{n+1} b$ iff there exists a $c \in I$ such that $a \sqsubseteq^n c$ and $b \in P(c)$;

  $a \sqsubseteq b$ iff there exists an $n \in \omega$ such that $a \sqsubseteq^n b$.

Thus, if $a \sqsubseteq b$, then a is a subset of b. A research is defined by Grzegorczyk as a structure $\mathcal{R} =< I, P, 1 >$,[4] where I is a set of information pieces (i.e. a set of finite sets of propositional variables), P is a mapping from I into $2^I - \{\emptyset\}$ satisfying (*), and every information piece is an extension of the *initial* information piece $1 \in I$: if $a \in I$, then $1 \sqsubseteq a$ (where $\sqsubseteq$ is defined as above). Ideally, $1 = \emptyset$. It can readily be seen that $< I, \sqsubseteq, 1 >$ is a tree. Next, for a given research $\mathcal{R} =< I, P, 1 >$, Grzegorczyk defines a basic valuation function $v_0 : PROP \cup \{\perp\} \longrightarrow 2^I$:

$$v_0(p) = \{a \in I \mid p \in a\}; \;\; v_0(\perp) = \emptyset.$$

The basic valuation function v_0 is inductively extended to a valuation function v from the set of all L-formulas into 2^I in exactly the same way as for Kripke models. Thus, every research $< I, P, 1 >$ can be regarded as an intuitionistic Kripke model $< I, \sqsubseteq_P, 1, v_0 >$ based on a tree. Validity of a sequent $s = A_1 \ldots A_n \rightarrow A$ in a research $\mathcal{R} =< I, P, 1 >$ is defined as the validity of s in $< I, \sqsubseteq_P, 1, v_0 >$. Grzegorczyk proves the following characterization theorem:

Theorem 1.15 A is a theorem of IPL iff A is valid in every research.

[4] Grzegorcyz uses '0' instead of '1'.

Completeness is proved by Grzegorczyk in an indirect way. He shows that every finite tree T induces a research $\mathcal{R}$ isomorphic to T such that for every L-formula A, A is valid on T according to the topological interpretation of intuitionistic propositional logic[5] iff A is valid in $\mathcal{R}$. Thus, if A is valid in every research $\mathcal{R}$, then it is valid on every finite tree according to the topological interpretation of IPL, and thus it is a theorem of IPL.

Grzegorczyk does not define a canonical research for IPL, and it can easily be shown that such a research doesn't exist. Suppose that $\mathcal{R}$ is a canonical research for IPL with the set of information pieces I. Note that the set $\Gamma = \bigcup\{a \mid a \in I\}$ is finite. Now, take any $q \in PROP$ such that $q \notin \Gamma$. Then $q \setminus p$ is valid in $\mathcal{R}$ for arbitrary p, although $\not\vdash_{IPL} \rightarrow q \setminus p$.[6] Thus $\mathcal{R}$ fails to be canonical.

1.4 The BHK interpretation of IPL

Let us conclude the review of IPL by presenting an interpretation in terms of *proofs*, viz. the so-called Brouwer-Heyting-Kolmogorov interpretation (BHK interpretation) of the intuitionistic connectives $\wedge$, $\vee$, $\supset$ and the falsum constant $\perp$.[7] To begin with we adopt Girard's [Girard, Lafont & Taylor 1989, p. 5] point of view that "by a *proof* we understand not the syntactic formal transcript, but the inherent object of which the written form gives only a shadowy reflection. We take the view that what we *write* as a proof is merely a description of something which is *already* a process in itself". From a foundational perspective, the explanatory power of the BHK interpretation depends, of course, on the explanatory power of the notions it makes use of, such as "construction", "transform", etc.[8] In this connection Troelstra and van Dalen [1988, p. 9] point out that "on a very 'classical' interpretation of construction and mapping ... [the interpretation justifies] the principles of two-valued (classical) logic". With these remarks in mind let us consider one recent formulation of the BHK interpretation of IPL.

[Troelstra & van Dalen 1988, p. 9]

(H1) A proof of $A \wedge B$ is given by presenting a proof of A and a proof of B.

(H2) A proof of $A \vee B$ is given by presenting either a proof of A or a proof of B (plus the stipulation that we want to regard the proof presented as evidence for $A \vee B$).

(H3) A proof of $A \supset B$[9] is a construction which permits us to transform any proof of A into a proof of B.

(H4) Absurdity $\perp$ (contradiction) has no proof; a proof of $\neg A$ is a construction which transforms any hypothetical proof of A into a proof of a contradiction.

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[5] A presentation of the topological semantics for IPL can e.g. be found in [van Dalen 1986].

[6] Grzegorczyk uses '$\supset$' instead of '/' and '\'.

[7] The question of what can be regarded as a proof of a primitive sentence represented by a propositional variable "depends on the particular discipline that is being considered" [López-Escobar 1972, p. 363].

[8] An very eloquent version of the BHK interpretation can be found in [Dragalin 1988, p. 2ff.].

[9] Troelstra and van Dalen use '$\rightarrow$' instead of '$\supset$'.

Generally, the BHK interpretation is regarded as a "natural semantics" [Troelstra & van Dalen 1988, p. 24] for IPL. According to Girard [1989, p. 71] "Heyting's *semantics of proofs*" even is "[o]ne of the greatest ideas in logic".

1.5 Appendix: Derivations in sequent calculi

Sequent calculi are 'meta-calculi'. A single conclusion sequent calculus acts on sequents $X \to A$, i.e. it manipulates expressions saying that a formula A is a syntactic consequence of a finite sequence of formula occurrences X. At this meta-level we have a syntactic consequence relation $\vdash$ between finite sequences S of sequent occurrences and single sequents. If $\mathcal{L}$ is a logic presented as a sequent calculus, then $\mathcal{D}_{\mathcal{L}}(\Pi, X \to A, S)$, "$\Pi$ is a derivation in $\mathcal{L}$ of $X \to A$ from S" is defined in a way induced by the rules of $\mathcal{L}$. As an example we here give the complete definition for IPL:

- $\mathcal{D}_{IPL}(A \to A, A \to A, <>)$.
- If $\mathcal{D}_{IPL}(\Pi_1, Y \to A, S_1)$ and $\mathcal{D}_{IPL}(\Pi_2, XAZ \to B, S_2)$, then $\mathcal{D}_{IPL}(\frac{\Pi_1\Pi_2}{XYZ \to B}, XYZ \to B, S_1S_2)$.
- $\mathcal{D}_{IPL}(X\bot Y \to A, X\bot Y \to A, <>)$.
- $\mathcal{D}_{IPL}(X \to \mathbf{t}, X \to \mathbf{t}, <>)$.
- $\mathcal{D}_{IPL}(\to \top, \to \top, <>)$.
- If $\mathcal{D}_{IPL}(\Pi, XY \to A, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{X\top Y \to A}, X\top Y \to A, S)$.
- If $\mathcal{D}_{IPL}(\Pi, XA \to B, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{X \to (B/A)}, X \to (B/A), S)$.
- If $\mathcal{D}_{IPL}(\Pi_1, Y \to A, S_1)$ and $\mathcal{D}_{IPL}(\Pi_2, XBZ \to C, S_2)$, then $\mathcal{D}_{IPL}(\frac{\Pi_1\Pi_2}{X(B/A)YZ \to C}, X(B/A)YZ \to C, S_1S_2)$.
- If $\mathcal{D}_{IPL}(\Pi, AX \to B, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{X \to (A\setminus B)}, X \to (A \setminus B), S)$.
- If $\mathcal{D}_{IPL}(\Pi_1, Y \to A, S_1)$ and $\mathcal{D}_{IPL}(\Pi_2, XBZ \to C, S_2)$, then $\mathcal{D}_{IPL}(\frac{\Pi_1\Pi_2}{XY(A\setminus B)Z \to C}, XY(A \setminus B)Z \to C, S_1S_2)$.
- If $\mathcal{D}_{IPL}(\Pi_1, X \to A, S_1)$ and $\mathcal{D}_{IPL}(\Pi_2, Y \to B, S_2)$, then $\mathcal{D}_{IPL}(\frac{\Pi_1\Pi_2}{XY \to (A\circ B)}, XY \to (A \circ B), S_1S_2)$.
- If $\mathcal{D}_{IPL}(\Pi, XABY \to C, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{X(A\circ B)Y \to C}, X(A \circ B)Y \to C, S)$.
- If $\mathcal{D}_{IPL}(\Pi_1, X \to A, S_1)$ and $\mathcal{D}_{IPL}(\Pi_2, X \to B, S_2)$, then $\mathcal{D}_{IPL}(\frac{\Pi_1\Pi_2}{X \to (A\wedge B)}, X \to (A \wedge B), S_1S_2)$.
- If $\mathcal{D}_{IPL}(\Pi, XAY \to C, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{X(A\wedge B)Y \to C}, X(A \wedge B)Y \to C, S)$.
  If $\mathcal{D}_{IPL}(\Pi, XBY \to C, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{X(A\wedge B)Y \to C}, X(A \wedge B)Y \to C, S)$.

- If $\mathcal{D}_{IPL}(\Pi, X \to A, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{X \to (A \vee B)}, X \to (A \vee B), S)$.
  If $\mathcal{D}_{IPL}(\Pi, X \to B, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{X \to (A \vee B)}, X \to (A \vee B), S)$.
- If $\mathcal{D}_{IPL}(\Pi_1, XAY \to C, S_1)$ and $\mathcal{D}_{IPL}(\Pi_2, XBY \to C, S_2)$, then $\mathcal{D}_{IPL}(\frac{\Pi_1 \Pi_2}{X(A \vee B)Y \to C}, X(A \vee B)Y \to C, S_1 S_2)$.
- If $\mathcal{D}_{IPL}(\Pi, XABY \to C, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{XBAY \to C}, XBAY \to C, S)$.
- If $\mathcal{D}_{IPL}(\Pi, XAAY \to B, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{XAY \to B}, XAY \to B, S)$.
- If $\mathcal{D}_{IPL}(\Pi, XY \to B, S)$, then $\mathcal{D}_{IPL}(\frac{\Pi}{XAY \to B}, XAY \to B, S)$.