Rearrangements of functions, variational problems and elliptic equations for vortices

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Summary

The results of my thesis concern the existence theory for certain variational problems, which are related to vortex flows of ideal fluids. The method that is used, is similar to the one that was proposed by Benjamin [5] and motivated recent work by Burton [10], in which vortex rings can be obtained as maximisers of a functional that is related to the kinetic energy over the set of rearrangements of a fixed function.

We start out in the first Chapter by explaining some preliminary concepts in our work (rearrangements of functions and existence theorems for steady vortices). We then explain Burton's method for maximisation of functionals and briefly state of the main results of this thesis.

In the second Chapter, we develop the existence theory of a variational problem similar to the one governing steady 2-dimensional ideal fluid flows containing symmetric vortex pairs. A functional related to the kinetic energy and the "generalised impulse" I_n corresponding to the parameter $n > 0$, is shown to attain a maximum value relative to the set of rearrangements of a prescribed function. Specifically, if λ is a parameter corresponding to the strength of the background flow at infinity, then we show that there exists a maximiser among flows whose vortices are rearrangements of a prescribed function, for all $\lambda > 0$ when $n \geq 3$ and for only small $\lambda > 0$ if $n = 2$, where this last case represents the existence of vortex pairs in a "two phase" shear flow.

In the third Chapter, we adapt the method of Burton [18] to study a problem related to the one studied in Chapter 2. We maximise a functional that is related to the kinetic energy over the weak closure of the set of rearrangements of a prescribed function, and for which the "generalised impulse" I_n has a prescribed value. We prove that if $n \geq 3$, then for any I_n, the constrained maximiser is a rearrangement of the prescribed function, and if $n \in \{1, 2\}$, then the maximiser is a rearrangement only for I_n large.

In the last Chapter, we prove the existence theory for a slightly different variational problem, governing a steady 3-dimensional ideal fluid flow containing axisymmetric steady vortex rings. The method that is used here is similar to that used in Chapter 2. If λ is the value of the strength of a steady ideal fluid flow at infinity, and I_{2n} is the "generalised impulse", then we prove that for all $\lambda > 0$ and $n \geq 4$, a functional related to the kinetic energy and the "generalised impulse" I_{2n}, is attained a maximum value relative to the set of rearrangement of a prescribed function. We also show that by making an assumption which we believe to be true, in the case when $n \in [2, 4[$ and λ is small, the same functional is attained a maximum value relative to set of rearrangements of prescribed function for all small $\lambda > 0$. The case when $n = 2$ represents the existence of a steady vortex ring in Poiseuille flow.

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