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Representations of Algebras

Tame and Wild Behavior

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To Nelia.
To the memory of Andrzej Skowroński and Daniel Simson, great mathematicians and great friends.

Preface

The representation theory of associative algebras has developed rapidly since the 1970s, perhaps due to the depth within the problems and conjectures posed early on in the theory, perhaps for the role these problems play in the machinery of mathematics. This book intends to collect some of the most relevant advances obtained surrounding the tame and wild dichotomy problem, and what are now known as Brauer-Thrall conjectures. With these problems as guideline, we consider topics as integral quadratic forms, lattices of ideals, Auslander-Reiten theory, Bass's theorems on (semi-) perfect rings, Galois coverings and smash products, Coxeter spectral analysis, Weyl groups, and post-projective components, among others, all applied to the representation theory of wide classes of associative algebras.

A part of my professional career has been dedicated to the understanding of some classes of algebras, from a representation theoretical perspective, in view of the problems mentioned above, which has undoubtedly biased the selection of topics of the text. The task has led me to collaborate with some of the most influential figures in the development of the theory, cherished collaborations that morphed into friendships over time.

Throughout the text, A will denote a finite dimensional K-algebra over an algebraically closed field K. The main reason for taking these hypotheses is simplicity. For instance, gaining a uniform context not depending on whether we consider left or right modules by means of a duality functor

$$D = \mathrm{Hom}_K(-, K) : A\text{−Mod} \to A^{op}\text{−Mod},$$

taking left modules to right modules and preserving length. Moreover, finite length modules over A are finite dimensional, and therefore, the category $A-$mod is contained in the category of vector spaces $K-$mod. Finally, the consideration of an algebraically closed field sets in advance a nice hypothesis that will be used in many instances (for dichotomy theorems, for the existence of quivers, for the existence of roots of unity and other situations).

We say that A is of *finite representation type* if there are only finitely many indecomposable A-modules (up to isomorphism). The algebra A is *tame* if for every number n, almost every indecomposable A-module of dimension n is isomorphic to a module belonging to a finite number of 1-parameter families, where a 1-parameter family is of the form $S \otimes_L -$ for S a simple module and L a finitely generated $K[x] - K[x]$-bimodule. Finally, A is *wild* if $\mod_A$ contains the representation theory of $K\langle x, y\rangle$, the free associative algebra in two indeterminates, that is, there is a faithful functor of the form $F = M \otimes_{K\langle x,y\rangle}$—which preserves isomorphism-classes. Drozd's theorem states that every finitely generated K-algebra is of one of these types.

The first explicit recognition that infinite representation type splits in two different classes arises in representations of groups: in 1954, Highman showed that the Klein group has infinitely many representations in characteristic 2 and Heller and Reiner classified them; in contrast, Krugljak showed in 1963 that solving the classification problem of groups of type (p, p) with $p \geq 3$ implies the classification of the representations of any group of the same characteristic, a task that was recognized as 'wild'. Donovan and Freislich conjectured at the middle of the 1970s that algebras split in *tame* and *wild* types, which was finally showed by Yuri Drozd in 1980.

The class of representation infinite algebras can be divided as follows: In 1957, J. Jans (then a student of Thrall) showed that a non-distributive algebra is strongly unbounded, i.e., that there exist infinitely many d such that there are infinitely many isomorphism classes of indecomposables of dimension d. Furthermore, he mentions two conjectures of Brauer and Thrall: The first says that A is representation-finite if there is a bound on the dimensions of indecomposables (BT1), and the second says that otherwise A is strongly unbounded (BT2).

The first conjecture was solved by Roiter [1] in 1968 by proving a somewhat stronger theorem. (Indeed, assume there are infinitely many isomorphism classes of indecomposable modules M_i, take the direct sum $M = \bigoplus_i M_i$. By Krull-Remak-Schmidt-Azumaya Theorem, this module M is not of finite type. Thus we obtain indecomposable modules of arbitrarily large finite length as submodules of this particular module M.) For the generalization of BT1 to artinian rings in 1972, Auslander invented almost split sequences. Jans proves BT2 in his paper for algebras that are not distributive. There is the long article [2] Nazarova and Roiter aiming at a proof of BT2, but the first complete proof was only given by Bautista in 1983. The proof of the second conjecture required some of the new concepts of representation theory introduced after 1968 and also an intensive study of representation-finite and distributive minimal representation-infinite algebras.

Integral quadratic forms, their roots and reflections, are classical instruments in representation theory, since the seminal work by Gabriel determining those hereditary basic algebras that are representation finite, via quiver algebras and their representations. In this case, the representation type of the algebra is determined by the positive-definiteness of its Euler quadratic form, an integral quadratic form containing some of the homological information of the algebra. Weaker arithmetical properties of the Tits form of an algebra, also known as geometrical form for its interpretation as (bounds of) dimensions of some varieties of modules associated with the algebra, have also been used as criterion to test finite-representation type and tameness of certain classes of algebras, most notoriously, strongly or weakly simply connected algebras (for instance, by Bongartz, Geiss, Brüstle, Skowroński and the author). For a more complete treatment of integral quadratic forms and their roots, the reader is referred to the book by Barot, Jiménez and the author [3].

In Chap. 1, we present some of the classical problems that triggered a rapid development in the representation theory of algebras: the Brauer-Thrall conjectures and Drozd's dichotomy theorem. After some preparatory concepts on quadratic forms, we introduce fundamental concepts and examples on algebras, categories, and algebraic varieties. Chapter 2 contains elementary (but fundamental) results in representation theory, namely the lemmas of Fitting, Harada-Sai, and Yoneda. We present Auslander's functorial approach on the existence of almost split sequences and use these results, together with Bass's characterization of perfect rings, to give a proof of the first Brauer-Thrall conjecture. Chapter 3 starts with the definition and characterization of distributive algebras, and some comments on other propositions under the label of Brauer-Thrall conjectures. The rest of the chapter treats the existence of post-projective components through an algorithmic approach and uses Tits quadratic form to determine the representation type of triangular algebras. We also give criteria to determine weak positivity and non-negative of integral quadratic forms. Chapter 4 deals with the Coxeter transformation associated with the Cartan matrix of a basic algebra. We relate Coxeter spectral properties to the representation theoretical structure of an algebra and seek conditions in the automorphism group of an algebra to determine spectral properties of the correspondent Coxeter transformation. Chapter 5 is dedicated to fundamental concepts on symmetries, automorphisms, and coverings of algebras, with applications to their representation theory. We analyze classical construction as pull-up and push-down functors, graded categories, and smash products, and relate such constructions to the representation type of wide classes of algebras. We start Chap. 6 with useful numerical properties of Dynkin and Euclidean graphs known as Vinberg's characterizations and apply them to study the structure of Auslander-Reiten components of algebras. We also study the Weyl group associated with graphs and derive some interesting characterizations of wild behavior in these groups. In Chap. 7, we study fundamental groups and a family of algebras known as (strongly) simply connected. We introduce their basic properties and propose a constructive characterization of such algebras, expanding on the so-called weakly separating families of coils. In Chap. 8, we turn our attention to classical ideas on the geometry of algebras and their module varieties, mainly concerning degenerations of algebras and how their representation type is affected with through topological constructions. Some final comments and historical remarks are collected at the end in Chap. 9.

It is our intention to bring forward some important aspects of representation theory that are usually not covered in classical introductions to the topic, such as the books of Auslander, Reiten, and Smalø [4], of Assem, Simson, and Skowroński [5], and of Gabriel and Roiter [6] (see also recent introductions by Barot [7] and Assem and Coelho [8]), or even in the homological or model theoretical treatments by Zimmermann [9] or Jensen and Lenzing [10], respectively. In this sense, this work is parallel to, for instance, the books of Ringel [11], of Simson and Skowroński [12], and of Erdmann and Holm [13]. The text is suitable for advanced undergraduate students wishing to deepen in the representation theory of associative algebras (for example, as a second course), and for young researchers trying to find a path among the vast literature in the area published in the last few decades.

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References

1. Roiter, A.V. *The unboundeness of the dimension of the indecomposable representations of algebras that have an infinite number of indecomposable representations*, Izv. Acad. Nauk SSSR, Ser. Mat., **32** (1968), 1275–82 (in Russian).
2. Nazarova, L.A. and Roiter, A.V. *Kategorielle Matrizen-Probleme und die Brauer-Thrall-Vermutung*, Mitt. Math. Sem. Giessen **115** (1975), 1–153.
3. Barot, M., Jiménez González, J.A. and de la Peña, J.A. *Quadratic Forms: Combinatorics and Numerical Results*, Algebra and Applications, Vol. **25** Springer Nature Switzerland AG 2018
4. Auslander, M. and Reiten, I. and Smalø, S. *Representation theory of Artin algebras*, Cambridge University Press **36** Cambridge Studies in Advanced Mathematics (1995)
5. Assem, I. and Simson, D. and Skowronski, A., *Elements of the representation theory of associative algebras*, Cambridge University Press (2006), London Mathematical Society Student Texts **65**
6. Gabriel, P. and Roiter, A., *Representations of finite-dimensional algebras*, London Mathematical Society, LNS **362**. Springer-Verlag Berlin Heidelberg. (1997)
7. Barot, M., *Introduction to the Representation Theory of Algebras*, Springer (2015)
8. Assem,I. and Coelho, F., *Basic Representation Theory of Algebras*. Graduate Texts in Mathematics, 283 (2020) Springer.
9. Zimmermann, A., *Representation theory: a homological algebra point of view*, Heidelberg, Springer, 2014
10. Jensen, C.U. and Lenzing, H., *Model theory and representation theory*. Lecture Notes in Mathematics 832. Springer-Verlag, 1980

11. Ringel, C.M. *Tame algebras and integral quadratic forms*, Springer LNM, **1099** (1984)
12. D. Simson and A. Skowroński, *Elements of the Representation Theory of Associative Algebras 2: Tubes and Concealed Algebras of Euclidean Type*. London Mathematical Society Student Texts, Vol. **71**, Cambridge University Press, Cambridge, 2007.
13. Erdmann, K. and Holm, T., *Algebras and Representation Theory*. Springer Undergraduate Mathematics Series, 2018

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