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Research Topics in Analysis, Volume I

Grounding Theory

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To the memory of our parents

Preface

The aim of this two-volume work is to present in a comprehensive way the main ingredients of some of the active research topics in nonlinear analysis and of its applications. The two volumes will equip young researchers with a good theoretical background and a taste of how these tools can be used to deal with more advanced studies on various topics and applications. The preparatory nature of Volume 1 is emphasized by the presence of relevant (and sometimes challenging) problems at the end of each chapter, which will help the reader digest and test his/her understanding of the materials. Some of the problems also provide additional notions and results. The flow of the materials is "linear" in the sense that each chapter sets the ground in order for the next ones to develop and the topics to expand naturally. In Volume 2, the chapters are more independent, and the reader can choose a particular chapter of interest to focus on without much difficulty.

Volume 1 presents the foundations of the subject and provides the necessary introductory materials for analysis in order for young researchers to approach the subject at the research level equipped with the necessary knowledge and background, with which they will be able to identify the areas where new contributions and advancements can be made and know the tools that need to be deployed. For this reason, the first three chapters are devoted to the presentation of the main aspects of topology, measure theory, and functional analysis. Our aim is to familiarize the reader with the basic background knowledge of these subjects without overwhelming them with details and topics which are not relevant to the study of modern nonlinear analysis and its applications. Our presentation focuses only on the essential aspects, which are used in the applications. Therefore, the reader will gain a good overview of the subjects without getting carried away in details that are not relevant. Chapter 4 continues in this spirit and is devoted to a comprehensive analysis of the main function spaces, which one encounters in applications. So, we present Lebesgue spaces (also with variable exponents), Lebesgue–Bochner spaces (needed in evaluation equations), Sobolev spaces (an indispensable tools in the study of boundary value problems, a modern treatment of partial differential equations), and spaces of measures (used in relaxation theory as well as in diverse applications such as optimal control, mathematical economics, and game theory).

Specifically, we have

In Chap. 1, we present a compact but comprehensive introduction of topology examined from the point of view of an analyst.

In Chap. 2, we do the same approach for the subject of measure theory. We also present how topology and measure theory interact. Such an interaction is very common in modern analysis and its applications. The theory of Young measures, which is important in calculus of variations and in optimal control theory, is a characteristic example of such an interaction of the two subjects.

Chapter 3 deals with functional analysis and gives a detailed account of Banach space theory and of the spaces of linear operators between them.

Chapter 4 deals with various function spaces that one encounters in applications. So, we cover Lebesgue spaces including those with variable exponents, Lebesgue–Bochner spaces (an indispensable tool in evolution equations), Sobolev spaces (the heart of the modern treatment of partial differential equations), and spaces of measures (important in probability theory and applications such as the transport theory).

In Chap. 5, we pass to more concrete subjects, which are closely related to applications and more difficult to find in compact form. The first such topic to be examined is that of "Multivalued Analysis" (set-valued functions). Set-valued maps were originally introduced to address the needs of subjects such as optimal control, optimization, mathematical economics, game theory, and calculus of variations. We will have a complete overview of the subjects, examining them from both topological and measure-theoretic viewpoints.

In Chap. 6, we present the smooth and nonsmooth calculus of general maps, with the latter developed in parallel and in symbiotic relation with multivalued analysis. Our treatment covers many aspects of the theory of convex analysis and of locally Lipschitz functions. The latter are central in the modern approach to the geometric measure theory and also provide a natural framework to extend convex analysis to nonconvex functions. Furthermore, Lipschitz functions are important since they are the counterpart of smooth functions in the study of equations in general metric spaces. We establish remarkable continuity properties that convex functions exhibit and then introduce the notion of "convex subdifferential" that extends the concept of derivative to nonsmooth convex functions and develops the corresponding calculus.

In Chap. 7, we introduce certain important classes of nonlinear operators such as compact and Fredholm operators, maximal monotone operators, and operators of monotone type. So, we deal with compact operators and with monotone operators and the generalizations. These classes arise naturally from nonsmooth analysis (Chap. 6), the link being the notion of the convex subdifferential. Compact operators are important because of their spectral properties and are the main ingredient in the infinite-dimensional extensions of degree theory (Leray–Schauder degree). On the other hand, operators of monotone type have many applications in nonlinear boundary value problems.

Complementing Chaps. 5–7, Chapter 8 presents some topics of variational analysis. So, we study convergence of sets, a topic useful in "Shape-Optimization" and "Sensitivity Analysis" of optimization and optimal control problems (a guide for the development of efficient numerical algorithms). Then we present the Γ-convergence of functions and the G-convergence of operators, two topics crucial in problems of the calculus of variations and of homogenization theory. Finally, we present some important variational principles such as the Ekeland variational principle, which are useful in applications.

Each chapter is followed by a collection of related exercise problems, which are of different levels of difficulty and are chosen so that the reader can check and confirm his understanding of the concepts introduced in the chapter. Also, in some occasions, new notions and results are introduced in the exercise problems.

The flow of the materials in this volume is natural in that each chapter leads naturally and smoothly to the next one. There are also listings of detailed indices of terminologies and mathematical symbols. We believe that the volume will equip the reader with necessary tools and theoretical background to address more specialized research topics and applications.

Volume 2 consists of various applications using tools and techniques developed in Volume 1. Volume 2 will be oriented to have the following eight chapters.

Chapter 1: "Degree Theory" The material of this chapter is useful to everyone applying the topological methods in the study of boundary value problems. So, we will present the Brouwer degree, the Leray–Schauder degree, and the degree theory for operator type and for multifunctions.

Chapter 2: "Fixed Point Theory" This is a topic used in all kinds of applications beyond mathematical. We will cover metric, topological, and order fixed point theories and also discuss the Fixed Point Index.

Chapter 3: "Critical Point Theory" This is the main tool in the variational methods in the study of boundary value problems. Together with critical groups (Morse theory), it leads to the powerful existence and multiplicity results for elliptic equations.

Chapter 4: "Spectra of Differential Operators" We will discuss about the spectrum under different boundary conditions of the Laplacian and the p-Laplacian including the scalar case, and of anisotropic and fractional ones.

Chapter 5: "Boundary Value Problems" In this chapter, we will examine concrete equations and will show how the material from the previous chapters can be used to obtain existence and multiplicity results for different classes of problems (variational, singular, problems with convection, anisotropic problems, etc.).

Chapter 6: "Evolution Equations" In this chapter, we will deal with dynamic equations and show how the formalism of evolution triples, and the theory of monotone operators can be used to treat certain parabolic and hyperbolic equations. We will also address the issue of the structure of the solution set and its stability using the notion of G-convergence.

Chapter 7: "Optimization and Optimal Control" We will study problems in calculus of variations and of optimal control, focusing on their relaxation properties and on sensitivity analysis.

Chapter 8: "Mathematical Economics and Game Theory" We will present both equilibrium and dynamic economic models and also discuss about Nash equilibrium.

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