

Lecture Notes in Mathematics 2318

Jean Deteix  
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# Numerical Methods for Mixed Finite Element Problems

Applications to Incompressible  
Materials and Contact Problems



Springer

# Lecture Notes in Mathematics

Volume 2318

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Jean Deteix • Thierno Diop • Michel Fortin

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This work was supported by Agence Nationale de la Recherche (ANR-18-IDEX-0001).

ISSN 0075-8434

ISSN 1617-9692 (electronic)

Lecture Notes in Mathematics

ISBN 978-3-031-12615-4

ISBN 978-3-031-12616-1 (eBook)

<https://doi.org/10.1007/978-3-031-12616-1>

Mathematics Subject Classification: 74S05, 65N22, 65F10, 65F08, 74B20, 74M15

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