Algebra and Applications

Pierre Cartier
Frédéric Patras

Classical Hopf Algebras and Their Applications

Springer

Algebra and Applications

Volume 29

Algebra and Applications aims to publish well-written and carefully refereed monographs with up-to-date expositions of research in all fields of algebra, including its classical impact on commutative and noncommutative algebraic and differential geometry, K-theory and algebraic topology, and further applications in related domains, such as number theory, homotopy and (co)homology theory through to discrete mathematics and mathematical physics.

Particular emphasis will be put on state-of-the-art topics such as rings of differential operators, Lie algebras and super-algebras, group rings and algebras, Kac-Moody theory, arithmetic algebraic geometry, Hopf algebras and quantum groups, as well as their applications within mathematics and beyond. Books dedicated to computational aspects of these topics will also be welcome.

Announcement (30 November 2020)

Alain Verschoren (1954-2020), Professor of Mathematics and Honorary Rector of the University of Antwerp, became an editor of the Algebra and Applications series in 2000. His contribution to the development of the series over two decades was pivotal. We, the Springer mathematics editorial staff and the editors of the series, mourn his passing and bear him in fond and grateful remembrance.

More information about this series at http://www.springer.com/series/6253

Pierre Cartier · Frédéric Patras

Classical Hopf Algebras and Their Applications

Springer

Pierre Cartier
Institut des Hautes Études Scientifiques
Bures-sur-Yvette, France

Frédéric Patras
Laboratoire J.A.Dieudonné
Université Côte d'Azur
Nice, France

ISSN 1572-5553 ISSN 2192-2950 (electronic)
Algebra and Applications
ISBN 978-3-030-77844-6 ISBN 978-3-030-77845-3 (eBook)
https://doi.org/10.1007/978-3-030-77845-3

Mathematics Subject Classification: 16T05, 16S30, 16T15, 16T30, 16W10

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

The present volume is dedicated to classical Hopf algebras and their applications. By classical Hopf algebras, we mean Hopf algebras as they first appeared in the works of Borel, Cartier, Hopf, and others in the 1940s and 50s: commutative or cocommutative Hopf algebras. The purpose of the book is twofold. It first of all offers a modern and systematic treatment of the structure theory of Hopf algebras, using the approach of natural operations. According to it, the best way to understand the structure of Hopf algebras is by means of their endomorphisms and their combinatorics. We therefore put weight on notions such as pseudo-coproducts, characteristic endomorphisms, descent algebras, or Lie idempotents, to quote a few. We have included in this treatment the case of enveloping algebras of pre-Lie algebras, extremely important in the recent literature, many interesting Lie algebras actually being the Lie algebras obtained by antisymmetrization of a pre-Lie product.

Second, the book surveys important application fields, explaining how Hopf algebras arise there, what problems they allow to address, and presenting the corresponding fundamental results. Each application field would require a textbook on its own, we have therefore limited our exposition to introducing the main ideas and accounting for the most fundamental results on which the use of Hopf algebras in the field are grounded.

The book should thus be useful as a general introduction and reference on classical Hopf algebras, their structure and endomorphisms; as a textbook for Master 2 or doctoral-level programs; and mostly and ultimately to scholars in algebra and the main application fields of Hopf algebras.

As for the book itself, it is the result of a long-lasting project. It originates ultimately in 1989, when one of the authors initiated a Ph.D. under the direction of the other. One of the ideas that emerged then was that the combinatorics of dilations underlying the theory of finite integration as appearing in Hilbert's third problem (on equidecomposability of polytopes under pasting and gluing operations) had far-reaching applications and generalizations. It extends, for example, to properties of the direct sum of the symmetric group algebras or the study of power maps on H-spaces. This lead to a purely combinatorial proof of structure theorems for graded connected cocommutative Hopf algebras around which the content of the central chapters of the first part of the book is organized.

At the time, the interest of the mathematical community for classical Hopf algebras was limited. A certain number of classical tools and structure results were available and were for the most part enough for the needs of applications, for example, in rational homotopy—the subdomain of algebraic topology where torsion phenomena are ignored. The situation evolved progressively, leading to the writing of the present book that brings together classical results, some of which go back to the 1950s, and recent advances under the unifying point of view of combinatorial structure results and techniques.

Many developments have contributed to the renewal of interest for classical Hopf algebras. In algebraic combinatorics, the works of Ch. Reutenauer, J.-Y. Thibon, and others generated again interest for the combinatorial theory of free Lie algebras, Lie idempotents, (noncommutative) representation theory of symmetric groups, and related objects. From the mid-1990s onward, the theme of combinatorial Hopf algebras, whose first idea can be traced back to Rota, gained momentum and grew steadily up to becoming one of the leading arguments of contemporary algebraic combinatorics.

Another line of development has several independent origins: deformation theory, differential calculus and differential geometry, numerical analysis and control, theoretical physics... It relates to the notion of pre-Lie algebras and to Hopf algebras of trees, forests, and diagrams. The notion of pre-Lie algebra dates back from the early 1960s (Gerstenhaber, Vinberg) and can even be found earlier in the work of Lazard. From the group and Lie theoretic point of view, which is also one of the Hopf algebras, a key step in the development of the theory of pre-Lie algebras is due to Agrachev and Gamkrelidze in the beginning of the 1980s. In hindsight, their work started to develop the extension to pre-Lie algebras of the combinatorial theory of Lie algebras and their enveloping algebras. However, the systematic development of the theory is much more recent. The work of Connes and Kreimer on Hopf algebras in perturbative quantum field theory around 2000 played here a particularly important role. They featured the role of pre-Lie algebras of trees and Feynman diagrams and their enveloping algebras in renormalization. Brouder rapidly connected their insights with methods and results in numerical analysis. Pre-Lie algebras and their enveloping (Hopf) algebras came to the forefront of researches on Hopf algebras and their applications. The recent surge of Hopf algebra techniques in stochastics (with rough paths, regularity structures) connects to this line of development.

In algebraic topology, homological algebra and related areas, where the very notion of Hopf algebra was born, the use of Hopf algebra techniques was classical since the 1940s. Besides in the study of topological groups, they appear, for example, in the study of loop spaces, algebras of operations such as Steenrod's or homology of Eilenberg–MacLane spaces. Pre-Lie algebras first appear in this context with the work of Gerstenhaber. They relate to the more general idea of brace operations that was introduced in the mid-1990s by Getzler, Gerstenhaber, and Voronov in the context of cochain complexes and the theory of operads. Here, the Hopf algebras at play have a particular structure: they are free or cofree as (co) associative (co)algebras (free or cofree (co)commutative when arising from pre-Lie algebras). This idea of Hopf algebras with extra structures proved also important, as those structures carry with themselves the existence of additional properties and operations.

Algebraic combinatorics and combinatorial Hopf algebras; algebraic topology, homological algebra, and operadic structures; pre-Lie algebras together with their many applications: we can give only very fragmentary indications about the many developments that occured during the last 30 years and have deeply reshaped the subject of classical Hopf algebras. We mention specifically these three lines of thought since they motivated various choices made in the writing of this book. We also wanted to point out with these examples the high level of activity surrounding the subject of Hopf algebras, which appears over and over as a central topic in contemporary mathematics.

Overall, the subject is too vast to be covered by a single textbook, we therefore had to make choices. The book is structured into two parts: general theory and applications. In the first part, we give a systematic account of the structure theory of commutative or cocommutative Hopf algebras with emphasis put on enveloping algebras of graded or complete Lie algebras and the dual polynomial Hopf algebras, mostly over a field of characteristic 0. The second part is dedicated to several key applications of the theory, classical, and recent. These application chapters can be read separately, but we advise the reader seriously interested in using Hopf algebras to read all of them as they offer complementary insights. Many techniques and intuitions can actually be carried over from an application field to another.

It is impossible to acknowledge here all those who contributed along the years by discussion, collaborations, and joint works to the building of the picture of Hopf algebras and their applications addressed hereafter.

Frédéric Patras would like to thank especially Kurusch Ebrahimi-Fard together with those others with whom he developed long-lasting research projects on the topics addressed in this book; many of these projects have run over the last 20 years and are still ongoing: Christian Brouder, Patrick Cassam-Chenaï, Loïc Foissy, Joachim Kock, Claudia Malvenuto, Simon Malham, Dominique Manchon, Frédéric Menous, Christophe Reutenauer, Nikolas Tapia, Jean-Yves Thibon, Anke Wiese, and Lorenzo Zambotti. A special thought to a late friend, Manfred Schocker: we had started together a vast program on Hopf algebras in combinatorics that was interrupted by his premature death, the chapter dedicated to combinatorial Hopf algebras is a tribute to his memory.

Limours, France Pierre Cartier
Nice, France Frédéric Patras

Conventions

All linear structures are defined over a ground field k, excepted otherwise specified.

Categories are written with bold symbols, for example, **Alg** stands for the category of algebras with a unit over k.

Algebras are algebras with unit, coalgebras are coalgebras with counit, excepted otherwise specified. Ideals are two-sided, that is, simultaneously left and right ideals, excepted otherwise specified.

We tend to abbreviate notations. For example, we will often write A for an algebra, instead of the triple (A, m_A, η_A), where m_A and η_A stand for the product and the unit.

Symbols

When a symbol (e.g., $\mathbf{Cog}_k$) is followed by "resp., ..." (e.g., resp., **Cog**), this means that the first symbol is the complete symbol associated to a notion, whereas the following ones stand for abbreviations used to alleviate the notation when no confusion can arise.

A^+: augmentation ideal of an augmented algebra A.

Abe: category of abelian groups.

$\mathbf{Alg}_k$ (resp., **Alg**): category of associative unital algebras.

$Aut_{\mathbf{C}}(X)$ (resp., $Aut(X)$): automorphisms of X in the category **C**.

$\mathbf{Alg}_k^c$ (resp., $\mathbf{Alg^c}$): category of complete augmented algebras.

$c^{(1)} \otimes \ldots \otimes c^{(n)}$: (abbreviated) Sweedler notation for $\Delta_n(c)$.

$\mathbb{C}$: complex numbers.

$\mathbf{C}(A, B)$: set of morphisms in the category **C** from A to B (also denoted $Hom_{\mathbf{C}}(A, B)$).

$C(M)$: subcoalgebra of a coalgebra C associated to a C-comodule M.

Cmd (resp., $\mathbf{Cmd}_C$): category of comodules over a coalgebra C.

$\mathbf{Cog}_k$ (resp., **Cog**): category of coassociative counital coalgebras.

Com$_k$ (resp., **Com**): category of commutative unital algebras.

δ_i^j: Kronecker delta function ($\delta_i^j = 1$ if $i = j$ and 0 else).

δ_M (resp., δ): coproduct for a comodule M.

δ_V: identity of V viewed as an element of $V \otimes V^*$.

δ_x: delta function ($\delta_x(y) := \delta_x^y$).

Δ_C, Δ_H (resp., Δ): coproduct of the coalgebra C, the Hopf algebra H...

Δ_d: deconcatenation coproduct.

Δ_u: unshuffle coproduct.

Δ_l: $l-1$-fold iteration from C to $C^{\otimes l}$ of a coassociative coproduct $\Delta : C \to C \otimes C$.

$\bar{\Delta}$: reduced coproduct associated to a coproduct Δ.

$*$: dual (V^* is the dual of V).

η_A (resp., η): unit map $k \to A$ of an algebra, coaugmentation $\eta_C : k \to C$ of a coalgebra.

$Enc(V) = V \otimes V^* = End^\vee(V^*)$: vector space of linear endomorphisms of a finite-dimensional vector space V viewed as a coalgebra.

$End^\vee(V) = V^* \otimes V$: dual coalgebra of the algebra $End(V) = V \otimes V^*$ of linear endomorphisms of a finite-dimensional vector space V.

$End_{\mathbf{C}}(X)$ (resp., $End(X)$): set of endomorphisms of X in $\mathbf{C}$.

ε_C, ε_A (resp., ε): counit map $C \to k$ of a coalgebra, augmentation $\varepsilon_A : A \to k$ of an algebra.

$f_{|X}$: restriction of a map f to X, a subset of the domain of f.

$f^{|A}$: when a vector space decomposes as $W = A \oplus B$ and f is a linear map from V to W, it denotes the composition of the projection from W to A along B with f. We call $f^{|A}$ the corestriction of f to A.

Fin: category whose objects are the finite (possibly empty) subsets of $\mathbb{N}^*$ and whose morphisms are the bijections.

$\Gamma(B)$: set (resp., group) of group-like elements of a coalgebra (resp., Hopf algebra) B.

$GL(n,k)$: n-th general linear group over k.

Grp: category of groups.

$Hom_{\mathbf{C}}(A,B)$: set of morphisms from A to B in the category $\mathbf{C}$.

Hop$_k$ (resp., **Hop**): category of Hopf algebras.

Hop$_k^c$ (resp., **Hop**$^{\mathbf{c}}$): category of complete Hopf algebras.

Id_C (resp., Id): identity map of an object C in a given category.

k: ground field.

kG: group algebra of the group G.

k^G: k-valued functions on G.

$k[V]$: space of polynomials over a vector space V.

L_xf: left translate of f, $L_xf(y) = f(xy)$.

Lie$_k$ (resp., **Lie**): category of Lie algebras.

Lin$_k$ (resp., **Lin**): category of vector spaces.

$\mathbf{Lin}_k^c$ (resp., $\mathbf{Lin^c}$): category of complete filtered vector spaces.

$\mathbf{Lin}_k^f$ (resp., $\mathbf{Lin^f}$): category of filtered vector spaces.

$\mathbf{Lin}_k^g$ (resp., $\mathbf{Lin^g}$): category of graded vector spaces.

m_A, m_H (resp., m): algebra product of the algebra A, the Hopf algebra H...

m_l: $(l-1)$-fold iterated product of an algebra, from $A^{\otimes l}$ to A.

$M^{\otimes n}$: n-th tensor power $M \otimes \ldots \otimes M$ of M.

$M_n(k)$: square matrices of size $n \times n$ over k.

$\mathbf{Mod}_A$: category of left modules over an algebra A.

Mon: category of monoids.

$[n] := \{1, \ldots, n\}$.

$\mathbb{N}$: nonnegative integers.

$\mathbb{N}^*$: positive integers.

ν_H (resp., ν): $\nu_H := \eta_H \circ \varepsilon_H$, unit map of the convolution algebra $End_{\mathbf{Lin}}(H)$ of linear endomorphisms of a Hopf algebra.

$[0] := \emptyset$.

$Ob(\mathbf{C})$: class of objects of the category $\mathbf{C}$.

$O(n,k)$: $n \times n$ orthogonal group.

$\prod_{i=1}^{N} a_i$: in an associative algebra, and denotes the ordered product $a_1 \cdot \ldots \cdot a_N$.

$Prim(C)$: set of primitive elements of a coaugmented coalgebra C.

$\mathbb{Q}$: rational numbers.

$R_y f$: right translate of f, $R_y f(x) = f(xy)$.

$R(G)$: representative functions on a monoid or a group.

$\mathbb{R}$: real numbers.

$\sqcup\!\sqcup$: shuffle product.

Set: category of sets.

S_n: n-th symmetric group.

Spe: category of vector species.

T: switch map ($T(x \otimes y) := y \otimes x$), also written $T_{C,D}$ when mapping $C \otimes D$ to $D \otimes C$.

$T(V)$: space of tensors $\bigoplus_{n \in \mathbb{N}} V^{\otimes n}$ over a vector space V.

$TS(V)$: space of symmetric tensors over a vector space V.

$\mathbb{Z}$: integers.

Contents

Part II Applications