

José Luis Cisneros-Molina
Dũng Tráng Lê
José Seade *Editors*

Handbook of Geometry and Topology of Singularities II

 Springer

Handbook of Geometry and Topology of Singularities II

José Luis Cisneros-Molina · Dũng Tráng Lê ·
José Seade
Editors

Handbook of Geometry and Topology of Singularities II

 Springer

Editors

José Luis Cisneros-Molina
Instituto de Matemáticas
Unidad Cuernavaca
Universidad Nacional Autónoma de México
Cuernavaca, Mexico

Dũng Tráng Lê
Centre de Mathématiques et Informatique
Université d'Aix-Marseille
Marseille, France

José Seade
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Mexico City, Mexico

ISBN 978-3-030-78023-4 ISBN 978-3-030-78024-1 (eBook)
<https://doi.org/10.1007/978-3-030-78024-1>

© Springer Nature Switzerland AG 2021

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, expressed or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

This Springer imprint is published by the registered company Springer Nature Switzerland AG
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

Preface

This is the second volume of the Handbook of the Geometry and Topology of Singularities, a subject which is ubiquitous in mathematics, appearing naturally in a wide range of different areas of knowledge. The scope of singularity theory is vast; its purpose is multifold. This is a meeting point where many areas of mathematics, and science in general, come together.

Let us recall Bernard Teissier's words in his foreword to the Handbook in Volume I:

I claim that Singularity Theory sits inside Mathematics much as Mathematics sits inside the general scientific culture. The general mathematical culture knows about the existence of Morse theory, parametrizations of curves, Bézout's theorem for plane projective curves, zeroes of vector fields and the Poincaré-Hopf theorem, catastrophe theory, sometimes a version of resolution of singularities, the existence of an entire world of commutative algebra, etc. But again, for the singularist, these and many others are lineaments of a single landscape and she or he is aware of its connectedness. Moreover, just as Mathematics does with science in general, singularity theory interacts energetically with the rest of Mathematics, if only because the closures of non-singular varieties in some ambient space or their projections to smaller dimensional spaces tend to present singularities, smooth functions on a compact manifold must have critical points, etc. But singularity theory is also, again in a role played by Mathematics in general science, a crucible where different types of mathematical problems interact and surprising connections are born.

The Handbook has the intention of covering a wide scope of singularity theory, presenting articles on various aspects of the theory and its interactions with other areas of mathematics in a reader-friendly way. The authors are world experts; the various articles deal with both classical material and modern developments.

Volume I of this collection gathered together ten articles with foundational aspects of the theory. These include

- The combinatorics and topology of plane curves and surface singularities.
- An introduction to four of the classical methods for studying the topology and geometry of singular spaces, namely resolution of singularities, deformation theory, stratifications, and slicing the spaces à la Lefschetz.

- Milnor fibrations and their monodromy.
- Morse theory for stratified spaces and constructible sheaves.
- Simple Lie algebras and simple singularities.

Volume II also consists of ten articles. These cover foundational aspects of the theory and related topics, including

- The analytic classification of plane curve singularities and the existence of complex and real algebraic curves in the plane with prescribed singularities.
- An introduction to the limits of tangents to a complex analytic surface, a subject that originates in Whitney's work on understanding the set of limits of tangents at smooth points as one approaches the singular set.
- Introductions to Zariski's equisingularity and Intersection homology, which are two of the main current viewpoints for studying singularities. Equisingularity means equivalent or similar singularity in some sense that has to be made precise and it is a vast field of current research. Intersection homology was introduced by Mark Goresky and Robert MacPherson and is a brilliant way of making the famous duality theorems for compact oriented manifolds work for singular varieties.
- An overview of Milnor's fibration theorem for real and complex singularities, as well as an introduction to Massey's theory of L \hat{e} cycles, which encode deep information about the geometry and topology of the Milnor fibers of complex hypersurface singularities.
- A discussion of mixed singularities, which are real analytic singularities with a rich structure that allows their study via complex geometry. This uses the method of the non-degenerate Newton boundary and toric modifications, which are powerful tools for the study of complex analytic singularity theory.
- The study of intersections of concentric ellipsoids in \mathbb{R}^n and its relation with several areas of mathematics, from holomorphic vector fields to singularity theory, toric varieties, and moment-angle manifolds.
- A review of the topology of quasi-projective varieties and generalizations of results about the topology of the complements of singular plane curves and hypersurfaces in projective space.

Each chapter in Volume II has its own introduction and a large bibliography for further reading, and there is a global index of terms at the end.

This collection, the Handbook of Geometry and Topology of Singularities, will continue with three more volumes. Volumes III and IV will include contributions on Zariski equisingularity; the basic theory of \mathcal{A} -equivalence and density of stable maps due to John Mather, Terry Wall, and others; various aspects of the theory of Chern classes for singular varieties; indices of vector fields, 1-forms and foliations, extending the classical local index of Poincaré-Hopf; Lipschitz geometry in singularity theory; an introduction to mixed Hodge structures; limits of tangent spaces in high dimensions; tropical geometry; determinantal varieties; constructible sheaves and other important aspects of singularity theory. Volume V will be devoted to singular holomorphic foliations in complex manifolds.

Re-phrasing Bernard Teissier's words above, these topics, among many others, are lineaments of the single landscape that goes under the name "singularity theory".

There is a lot more that ought to be included in this collection but, happily, the vastness of this rich area of mathematics makes impossible the task of gathering together in five volumes so many important aspects. Yet, these five volumes together will cover a wide spectrum of singularity theory and its interactions with other related areas of mathematics.

This book is addressed to graduate students and newcomers to the theory, as well as to specialists who can use it as a guidebook, and it provides an accessible account of the state of the art in several aspects of the subject, its frontiers, and its interactions with other areas of research.

Cuernavaca, Mexico
Marseille, France
Mexico City, Mexico
March 2021

José Luis Cisneros-Molina
Dũng Tráng Lê
José Seade

Contents

| | | |
|-----------|---|------------|
| 1 | The Analytic Classification of Irreducible Plane Curve Singularities | 1 |
| | Abramo Hefez and Marcelo Escudeiro Hernandes | |
| 2 | Plane Algebraic Curves with Prescribed Singularities | 67 |
| | Gert-Martin Greuel and Eugenii Shustin | |
| 3 | Limit of Tangents on Complex Surfaces | 123 |
| | Tráng Dũng Lê and Jawad Snoussi | |
| 4 | Algebro-Geometric Equisingularity of Zariski | 177 |
| | Adam Parusiński | |
| 5 | Intersection Homology | 223 |
| | Jean-Paul Brasselet | |
| 6 | Milnor’s Fibration Theorem for Real and Complex Singularities | 309 |
| | José Luis Cisneros-Molina and José Seade | |
| 7 | Lê Cycles and Numbers of Hypersurface Singularities | 361 |
| | David B. Massey | |
| 8 | Introduction to Mixed Hypersurface Singularity | 403 |
| | Mutsuo Oka | |
| 9 | From Singularities to Polyhedral Products | 463 |
| | Santiago López de Medrano | |
| 10 | Complements to Ample Divisors and Singularities | 501 |
| | Anatoly Libgober | |
| | Index | 569 |