N. Hayashi · E.I. Kaikina · P.I. Naumkin
I.A Shishmarev

Asymptotics for Dissipative Nonlinear Equations

Springer

Authors

Nakao Hayashi
Department of Mathematics
Graduate School of Science
Osaka University, Osaka
Toyonaka 560-0043
Japan
e-mail: nhayashi@math.wani.osaka-u.ac.jp

Pavel I. Naumkin
Instituto de Matemáticas
UNAM Campus Morelia
AP 61-3 (Xangari)
Morelia CP 58089, Michoacán
Mexico
e-mail: pavelni@matmor.unam.mx

Elena I. Kaikina
Instituto de Matemáticas
UNAM Campus Morelia
AP 61-3 (Xangari)
Morelia CP 58089, Michoacán
Mexico
e-mail: ekaikina@matmor.unam.mx

Ilya A. Shishmarev
Department of Computational
Mathematics and Cybernetics
Moscow State University
Moscow 119 899
Russia
e-mail: shish@voxnet.ru

Library of Congress Control Number: 2006921737

Mathematics Subject Classification (2000): 35Axx, 35Bxx, 35Qxx, 35Sxx, 45XX, 76XX

ISSN print edition: 0075-8434
ISSN electronic edition: 1617-9692
ISBN-10 3-540-32059-8 Springer Berlin Heidelberg New York
ISBN-13 978-3-540-32059-3 Springer Berlin Heidelberg New York

DOI 10.1007/b133345

Springer is a part of Springer Science+Business Media
springer.com

Printed in The Netherlands

Typesetting: by the authors and SPI Publisher Services using a Springer LaTeX package
Cover design: *design & production* GmbH, Heidelberg

Printed on acid-free paper SPIN: 11665311 VA41/3100/SPI 5 4 3 2 1 0

Preface

Modern mathematical physics is almost exclusively a mathematical theory of nonlinear partial differential equations describing various physical processes. Since only a few partial differential equations have succeeded in being solved explicitly, different qualitative methods play a very important role. One of the most effective ways of qualitative analysis of differential equations are asymptotic methods, which enable us to obtain an explicit approximate representation for solutions with respect to a large parameter time. Asymptotic formulas allow us to know such basic properties of solutions as how solutions grow or decay in different regions, where solutions are monotonous and where they oscillate, which information about initial data is preserved in the asymptotic representation of the solution after large time, and so on. It is interesting to study the influence of the nonlinear term in the asymptotic behavior of solutions. For example, compared with the corresponding linear case, the solutions of the nonlinear problem can obtain rapid oscillations, can converge to a self-similar profile, can grow or decay faster, and so on. It is very difficult to obtain this information via numerical experiments. Thus asymptotic methods are important not only from the theoretical point of view, but also they are widely used in practice as a complement to numerical methods. It is worth mentioning that in practice large time could be a rather bounded value, which is sufficient for all the transitional processes caused by the initial perturbations in the system to happen.

The theory of asymptotic methods for nonlinear evolutionary equations is relatively young and traditional questions of general theory are far from being answered. A description of the large time asymptotic behavior of solutions of nonlinear evolution equations requires principally new approaches and the reorientation of points of view in the asymptotic methods. For example, the requirements of the infinite differentiability and a compact support usually acceptable in the linear theory are too strong in the nonlinear theory.

Asymptotic theory is difficult even in the case of linear evolutionary equations (see books Dix [1997], Fedoryuk [1999]). The difficulty of the asymptotic methods is explained by the fact that they need not only a global existence of solutions, but also a number of additional a priori estimates of the difference between the solution and the approximate solution (usually in the weighted norms). Also the generalized solutions could not be acceptable for the asymptotic theory, so we consider classical and semiclassical (mild) solutions, belonging to some Lebesgue spaces. Moreover in the case of nonlinear equations it is necessary to prove global existence of classical solutions and to obtain some additional estimates to clarify the asymptotic expansions. Every type of nonlinearity should be studied individually, especially in the case of large initial data.

A great number of publications have dealt with asymptotic representations of solutions to the Cauchy problem for nonlinear evolution equations in the last twenty years. While not attempting to provide a complete review of these publications, we do list some known results: Amick et al. [1989], Biler [1984], Biler et al. [1998], Biler et al. [2000], Bona et al. [1999], Bona and Luo [2001], Dix [1991], Dix [1992], Escobedo et al. [1995], Escobedo et al. [1993a], Galaktionov et al. [1985], Giga and Kambe [1988], Gmira and Véron [1984], Hayashi et al. [2001], Il′in and Oleĭnik [1960], Karch [1999a], Kavian [1987], Naumkin and Shishmarev [1989], Naumkin and Shishmarev [1990], Naumkin and Shishmarev [1994b], Schonbek [1986], Schonbek [1991], Strauss [1981], Zhang [2001], Zuazua [1993], Zuazua [1994], where, in particular, the optimal time decay estimates and asymptotic formulas of solutions to different nonlinear local and nonlocal dissipative equations were obtained. In the case of dispersive equations some progress in the asymptotic methods was achieved due to the discovery of the Inverse Scattering Transform method (see books Ablowitz and Segur [1981], Novikov et al. [1984] and papers Deift et al. [1993], Deift et al. [1994]). Some other functional analytic methods were applied for the study of the large time asymptotic behavior of solutions to dispersive equations in Cazenave [2003], Christ and Weinstein [1991], Clément and Nohel [1981], Georgiev and Milani [1998], Giga and Kambe [1988], Ginibre and Ozawa [1993], Glassey [1973a], Kenig et al. [2000], Kenig et al. [1997], Kita and Ozawa [2005], Kita and Wada [2002], Klainerman [1982], Klainerman and Ponce [1983], Ozawa [1995], Ozawa [1991], Ponce and Vega [1990], Segal [1968], Shimomura and Tonegawa [2004], Strauss [1974], Strichartz [1977], Tsutsumi and Hayashi [1984], Tsutsumi [1994].

This book is the first attempt to give a systematic approach for obtaining the large time asymptotic representations of solutions to the nonlinear evolution equations with dissipation. We restrict our attention to the investigation of the Cauchy problems (initial value problems) leaving outside the wide and important class of the initial-boundary value problems (in some respects the reader can fill this gap by consulting a recent book Hayashi and Kaikina [2004]). In our book we pay much attention to typical well-known equations which have huge applications: the nonlinear heat equation, Burgers equation, Korteweg-de Vries-Burgers equation, nonlinear damped wave equation, Landau-Ginzburg equation, Sobolev type equations, systems of equations of Boussinesq, Navier-Stokes equations and others. Certainly we do not claim that we could embrace all equations and all cases. However we succeeded in selecting a sufficiently wide class of equations, which could be treated by a unified approach and which fall into the same theory. Many of the methods proposed in this book have been developed by a great number of authors. The results and proofs presented throughout the book are mainly based on the research articles of the authors.

We divide nonlinear equations into three general types: asymptotically weak nonlinearity, critical nonlinearity and strong (or subcritical) nonlinearity. Also the critical and subcritical nonlinearities are divided into convective type and nonconvective type. In many cases nonlinearity leads to the blow-up of solutions in a finite time, so to be able to study global solutions we have to restrict our attention to small initial data. However we also closely examine the large time asymptotics for initial data of arbitrary size (not small).

Let us explain our classification taking the nonlinear heat equation as an example:

$$u_t - \Delta u + u^{\rho} = 0.$$

We assume that initial data are most general; however, in applications, the initial data are usually considered as sufficiently rapidly decaying at infinity and smooth, for example initial data $u_0 \in \mathbf{C}_0^{\infty}(\mathbf{R}^n)$ are acceptable. The decay rate at infinity of the initial data u_0 appears very essential, since for example taking $u_0 = C$ with a constant C, we then obtain the solution $u(t,x) = C$ of the Cauchy problem for the linear heat equation $u_t - \Delta u = 0$, so the solution does not decay at all. Physically this situation is slightly special, since usually some physical quantities, such as energy, mass and momentum, expressed via integrals of the solution, should be finite. Therefore we are interested when the initial data are integrable $u_0 \in \mathbf{L}^1(\mathbf{R}^n)$. Another special case occurs when the total mass of the initial data vanishes: $\int_{\mathbf{R}^n} u_0(x)\,dx = 0$. Then the solution of the linear heat equation $u_t - \Delta u = 0$ obtains some more rapid decay rate with time. In this case the critical value of the order of the nonlinearity is changed.

Now let us give a heuristic classification of the nonlinearities. Consider initial data $u_0(x) \in \mathbf{L}^1(\mathbf{R}^n)$ with nonzero total mass $\int_{\mathbf{R}^n} u_0(x)\,dx \neq 0$. Then the solution $u(t,x)$ of the Cauchy problem for the linear heat equation $u_t - \Delta u = 0$ has the following time decay rate $u(t,x) \sim t^{-\frac{n}{2}}$. If we compute the decay rate of the linear part of the equation we get $u_t - \Delta u \sim t^{-\frac{n}{2}-1}$. For such behavior of the solution the nonlinearity u^{ρ} decays as $u^{\rho} \sim t^{-\frac{n}{2}\rho}$ for $t \to \infty$. Now we see that if $\rho > 1 + \frac{2}{n}$, then the nonlinear term decays more rapidly than the linear part as time goes to infinity. We can expect that in this case the large time behavior of solutions is similar to the linear one, and it is possible to apply the results of the well-developed linear theory. We call $\rho > 1+\frac{2}{n}$ supercritical and the nonlinearity asymptotically weak. Respectively, cases $\rho = 1 + \frac{2}{n}$ and $\rho < 1 + \frac{2}{n}$ we call critical and subcritical. In the critical case there is a kind of equilibrium in the large time asymptotic behavior of linear and nonlinear parts of the equation. In the subcritical case as well, the nonlinear effects win in the large time asymptotic behavior of solutions. Hence the particular form of the nonlinearity is very important to determine the large time asymptotics of solutions in the critical and subcritical cases.

We found that there are two types of nonlinearity which define different asymptotic behavior of solutions in the critical and subcritical cases. By the convective type we mean the nonlinearity $\mathcal{N}(u)$ such that $f(\mathcal{N}(u)) = 0$, where the linear functional f determines the large time asymptotics of the corresponding linear problem (see Definition 2.1 for details). For example, the Burgers equation $u_t - u_{xx} + uu_x = 0$ has this type of the nonlinearity if the total mass of the initial data is nonzero. Then in the critical case the large time asymptotics is described by the self-similar solution and in the subcritical case the asymptotics of solutions is represented as a product of a rarefaction wave and a shock wave. Another asymptotic behavior occurs for the nonlinearities of nonconvective type, as in the nonlinear heat equation $u_t - \Delta u + u^\rho = 0$. In the critical case $\rho = 1 + \frac{2}{n}$ solutions have some additional logarithmic decay rate compared with the corresponding linear heat equation. In the subcritical case $1 < \rho < 1 + \frac{2}{n}$ solutions asymptotically approach special self-similar solutions also. Our method for critical and subcritical nonconvective equations is based on a change of the dependent variable such that the nonlinear term $\widetilde{\mathcal{N}}(u)$ of a modified equation has the property $f\left(\widetilde{\mathcal{N}}(u)\right) = 0$. Then the solutions of this new nonlinear equation have the asymptotic properties similar to that for the supercritical or critical convective equations. In the case of subcritical convective equations this method does not work, since the functional $f(\mathcal{N}(u))$ is already zero. Thus we develop another approach representing solutions in the form of the rarefaction and shock waves.

We would like to thank Professors T. Ozawa and K. Kato for many fruitful discussions on the subject of this book. Also we are grateful to unknown referees for many useful suggestions and comments.

Osaka, Morelia, Moscow
November, 2005

Nakao Hayashi
Elena I. Kaikina
Pavel I. Naumkin
Ilya A. Shishmarev

Contents