Horst Herrlich

# Axiom of Choice

Springer

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Dedicated in friendship
to George, Gerhard, and Lamar

*It is a peculiar fact that all the transfinite axioms are deducible from a single one, the axiom of choice, — the most challenged axiom in the mathematical literature.*

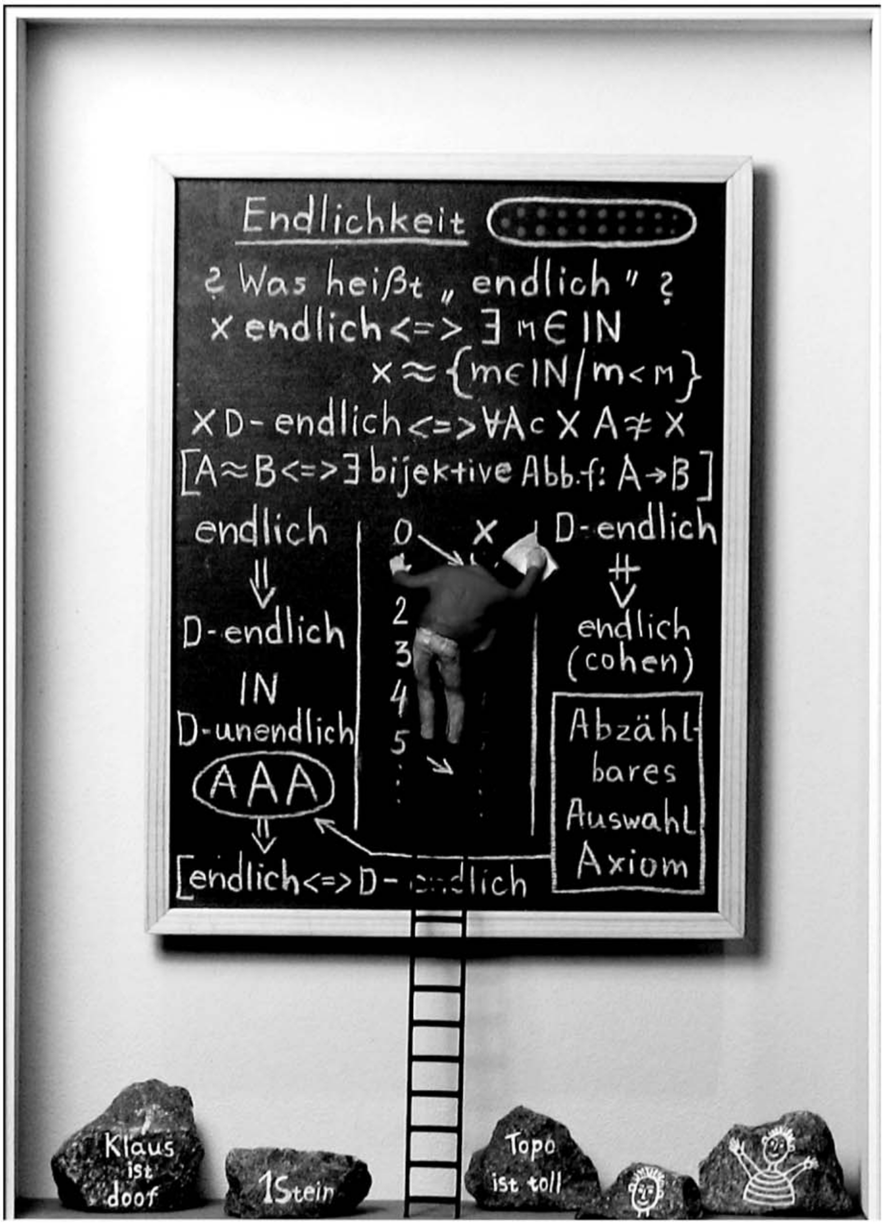
D. Hilbert (1926)

*It is the great and ancient **problem of existence** that underlies the whole controversy about the axiom of choice.*

W. Sierpiński (1958)

*Wie die mathematische Analysis gewissermaßen eine einzige Symphonie des Unendlichen ist.*

D. Hilbert (1926)




*Der Mathematiker.* Courtesy of the painter Volker Kühn (www.artinboxes.com).

# Preface

*Zermelo's proof, and especially the Axiom of Choice on which it was based, created a furor in the international mathematical community.*
...
*The Axiom of Choice has easily the most tortured history of all the set–theoretic axioms.*

Penelope Maddy (Believing the axioms I)[1]

*Of course not, but I am told it works even if you don't believe in it.*

Niels Bohr (when asked whether he really believed a horseshoe hanging over his door would bring him luck).[2]

Without question, the Axiom of Choice, **AC** (which states that for every family of non–empty sets the associated product is non–empty[3]), is the most controversial axiom in mathematics. Constructivists shun it, since it asserts the existence of rather elusive non–constructive entities. But the class of critics is much wider and includes such luminaries as J.E. Littlewood and B. Russell who objected to the fact that several of its consequences such as the Banach–Tarski Paradox are extremely counterintuitive, and who claimed that *"reflection makes the intuition of its truth doubtful, analysing it into prejudices derived from the finite case"*[4], resp. that *"the apparent evidence of the*

1 [Mad88]
2 c. 1930. Cited from: The Oxford Dictionary of Modern Quotations. Second Edition with updated supplement. 2004.
3 cf. Definition 1.1.
4 [Lit26]

*axiom tends to dissipate upon the influence of reflection"*[5]. (See also the comments after Theorem 1.4.) Nevertheless, over the years the proponents of **AC** seemed to have won the debate, first of all due to the fact that disasters happen without **AC**: many beautiful theorems are no longer provable, and secondly, Gödel showed that **AC** is relatively consistent[6]. So **AC** could not be responsible for any antinomies which might emerge. This somewhat opportunistic attitude, sometimes supported by such arguments as *"Even if we knew that it was impossible ever to define a single member of a class, it would not of course follow that members of the class did not exist."*[7], led to the situation that in most modern textbooks **AC** is assumed to be valid indiscriminately. Still, these facts only show the usefulness of **AC** not its validity, and Lusin's verdict[8] *"For me the proof of a theorem by means of Zermelo's axiom is valuable only as an indication that it is useless to waste time on an exact proof of the falsity of the theorem in question"* is still shared at least by the constructivists. Unfortunately, our intuition is too hazy for considering **AC** to be *evidently* true or *evidently* false, as expressed whimsically by J.L. Bona: "*The Axiom of Choice is obviously true, the Well–Ordering Principle is obviously false; and who can tell about Zorn's Lemma"*.[9]

Observe however that the distinction between the Axiom of Choice and the Well–Ordering Theorem is regarded by some, e.g. by H. Poincaré, as a serious one:

> *"The negative attitude of most intuitionists, because of the existential character of the axiom* [of choice]*, will be stressed in Chapter IV. To be sure, there are a few exceptions, for the equivalence of the axiom to the well–ordering theorem (which is rejected by all intuitionists) depends, inter alia, on procedures of a supposedly impredicative character; hence the possibility exists of accepting the axiom but rejecting well–ordering as it involves impredicative procedures. This was the attitude of Poincaré."*[10]

When Paul Cohen demonstrated that the negation of **AC** is relatively consistent too[11], and when he created a method for constructing models of **ZF** (i.e., Zermelo–Fraenkel set theory without the Axiom of Choice) in which not only **AC** fails, but in which certain given substitutes of **AC** — either weakening **AC** or even contradicting **AC** — hold, he triggered *"the post Paul*

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5 [Rus11]
6 [Goed39]
7 [Hard06]
8 Lusin 1926, cited after [Sie58, p. 95].
9 [Sch97, p. 145]
10 [FrBaLe73, p. 81]
11 [Coh63/64]

*Cohen set–theoretic renaissance"*[12], and a vast literature emerged in which **AC** is not assumed; thus giving life to Sierpiński's program[13]:

> *"Still, apart from our being personally inclined to accept the axiom of choice, we must take into consideration, in any case, its role in the Set Theory and in the Calculus. On the one hand, since the axiom of choice has been questioned by some mathematicians, it is important to know which theorems are proved with its aid and to realize the exact point at which the proof has been based on the axiom of choice; for it has frequently happened that various authors had made use of the axiom of choice in their proofs without being aware of it. And after all, even if no one questioned the axiom of choice, it would not be without interest to investigate which proofs are based on it and which theorems can be proved without its aid.*
> ...
> *It is most desirable to distinguish between theorems which can be proved without the aid of the axiom of choice and those which we are not able to prove without the aid of this axiom.*
> *Analysing proofs based on the axiom of choice we can*
> 1. *ascertain that the proof in question makes use of a certain particular case of the axiom of choice,*
> 2. *determine the particular case of the axiom of choice which is sufficient for the proof of the theorem in question, and the case which is necessary for the proof ...*
> 3. *determine that particular case of the axiom of choice which is both necessary and sufficient for the proof of the theorem in question."*

This book is written in Sierpiński's spirit, but one more step will be added which occurred neither to Sierpiński nor to Lusin, but was made possible by Cohen's work that opened new doors for set theorists: *"Set theory entered its modern era in the early 1960's on the heels of Cohen's discovery of the method of forcing and Scott's discovery of the relationship between large cardinal axioms and constructible sets."*[14] Some striking theorems will be presented, that can be proved to be false in **ZFC** (i.e., Zermelo–Fraenkel set theory with the Axiom of Choice), but which hold in **ZF** provided **AC** is replaced by some (relatively consistent) alternative axiom.

This book is not written as a compendium, or a textbook, or a history of the subject — far more comprehensive treatments of specific aspects can be found in the list of Selected Books and Longer Articles. I hope, however, that this monograph might find its way into seminars. Its purpose is to whet the

[12] J.M. Plotkin in the Zentralblatt review Zbl. 0582.03033 of [RuRu85].
[13] [Sie58, p. 90 and 96] Cf. also [Sie18]
[14] [Kle77]

reader's appetite for studying the **ZF**–universe in its fullness, and not just its highly interesting but rather small **ZFC**–part. Mathematics is sometimes compared with a cathedral, the mathematicians being simultaneously its architects and its admirers. Why visit only one of it wings — the one built with the help of **AC**? Beauty and excitement can be found in other parts as well — and there is no law that prevents those who visit one of its parts from visiting other parts, too.

An attempt has been made to keep the material treated as simple and elementary as possible. In particular no special knowledge of axiomatic set theory is required. However, a certain mathematical maturity and a basic acquaintance with general topology will turn out to be helpful.

The sections can be studied more or less independently of each other. However, it is recommended not to skip any of the sections 2.1, 2.2, or 3.3 since they contain several basic definitions.

A treatise like this one does not come out of the blue. It rests on the work of many people. Acknowledgments are due and happily given:

- to all those mathematicians — living or dead — whose work I have cannibalized freely, most of all to Paul Howard and Jean Rubin for their wonderful book, Consequences of the Axiom of Choice,
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Let us end the preface with the following three quotes:

*"Pudding and pie,"*
*Said Jane, "O, my!"*
*"Which would you rather?"*
*Said her father.*
*"Both," cried Jane,*
*Quite bold and plain.*
Anonymous (ca. 1907)

*The Axiom of Choice and its negation cannot coexist in one proof, but they can certainly coexist in one mind. It may be convenient to accept* **AC** *on some days — e.g., for compactness arguments — and to accept some alternative reality, such as* **ZF** + **DC** + **BP**[15] *on other days — e.g., for thinking about complete metric spaces.*

E. Schechter (1997)[16]

*So you see!*
*There's no end*
*To the things you might know,*
*Depending how far beyond Zebra you go!*
Dr. Seuss (1955)[17]

15 **DC** is the *Principle of Dependent Choices*; see Definition 2.11.
**BP** stipulates that every subset of $\mathbb{R}$ has the *Baire property*, i.e., can be expressed as a symmetric difference of an open set and a meager set; see [Sch97].
16 [Sch97]
17 From *On Beyond Zebra.*

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