Séminaires IRIA

ANALYSE ET CONTRÔLE DE SYSTÈMES

1977

IRIA

INSTITUT DE RECHERCHE D'INFORMATIQUE ET D'AUTOMATIQUE
DOMAINE DE VOLUCEAU - ROCQUENCOURT - B.P. 105 - 78150 LE CHESNAY - TÉL.: 954 90 20

ANALYSE ET CONTRÔLE DE SYSTÈMES

TEXTES DES EXPOSÉS DU SÉMINAIRE ORGANISÉ PAR
L'INSTITUT DE RECHERCHE D'INFORMATIQUE ET D'AUTOMATIQUE
(IRIA)
ROCQUENCOURT

1977

INSTITUT DE RECHERCHE D'INFORMATIQUE ET D'AUTOMATIQUE
DOMAINE DE VOLUCEAU - ROCQUENCOURT - B.P.105 - 78150 LE CHESNAY - TÉL.: 954 90 20

Édité par l'Institut de Recherche d'Informatique et d'Automatique
Dépôt légal 310378/200
I.S.B.N. 2 - 7261 - 0158 - 5

TABLE DES MATIÈRES

INVARIANT THEORY, THE RICCATI GROUP, AND LINEAR CONTROL PROBLEMS

J. CASTI

New York University, U.S.A.

Abstract

The classical algebraic theory of invariants is applied to the linear-quadratic-gaussian control problem to derive a canonical form under a certain matrix transformation group. The particular group of transformations, termed here the "Riccati group," is induced from the matrix Riccati equation characterizing the LQG problem solution.

Examples of the invariant-theoretic approach are given along with a discussion of topics meriting further study, including geometric interpretation of the group orbits, extension of the Riccati group, and connections with the generalized X-Y functions.

1. Introduction

One of the principal themes of nineteenth-century mathematics was the theory of invariants of transformation groups. This theory, developed to a high degree of sophistication by Hilbert, Cayley, Sylvester, Gordan, Dickson, and other, has been more or less relegated to the backwaters of mathematical activity until rather recently. A fascinating account of the rise and fall of the theory from the sociological point of view is given in [1,2].

In connection with renewed interest in invariant theory on the part of modern algebraists, there has been a corresponding interest in the use of invariant theoretic-techniques to shed light on certain algebraic issues arising in system theory. Since 1970, a number of papers have appeared [cf. 3-7] in which the general techniques developed by the nineteenth-century algebraists have been employed to characterize certain minimal-parameter canonical forms for constant, finite-dimensional linear systems. This work has been an outgrowth of one of the central themes of current research in mathematical system theory, namely, the use of modern algebra and differential geometry as tools to explore system-theoretic issues. This point of view, pioneered by Kalman, Hermann, Arbib, Brockett, and others, has developed from an offshoot of mathematical control and filtering theory to a legitimate branch of applied mathematics during the past decade. A good account of the current state of affairs may be found in [8].

Broadly speaking, past use of invariant theory in linear system problems has been confined to the following general set-up: a system Σ is given by the dynamics

$$\dot{x} = Fx + Gu \quad ,$$

and the output

$$y(t) = Hx \quad ,$$

where x, u, and y are n, m, and p-dimensional vectors, respectively with F, G, H being constant matrices of appropriate sizes. Thus, Σ is characterized by the matrix triple (F,G,H). In addition, a certain group $\mathcal{G}$ of transformations is given which acts on the triple (F,G,H) to produce a new system $(\bar{F},\bar{G},\bar{H})$, i.e.,

$$\Sigma = (F,G,H) \xrightarrow{\mathcal{G}} (\bar{F},\bar{G},\bar{H}) = \bar{\Sigma} \quad .$$

In essence (details later), the theory of invariants is used to find a set of polynomials in the components of F, G, H which remain "invariant" under application of transformations from $\mathcal{G}$ to Σ. A principle objective is to use the transformations from $\mathcal{G}$ in such a way that $\bar{\Sigma}$ assumes a "simpler" form than Σ. Usually, "simpler" has been interpreted to mean that $\bar{\Sigma}$ contains a minimal number of parameters, consistent with the group $\mathcal{G}$.

In the classical terminology, a collection of polynomials $\{p_\alpha\}$, $\alpha \in A$ (A an index set), forming a set of invariants under $\mathcal{G}$ is called:

i) independent if no algebraic relations exist among the polynomials;

ii) complete if $p_\alpha(q) = p_\alpha(q')$ for all $\alpha \in A$ implies $q = gq'$ for some $g \in \mathcal{G}$.

The basic work cited above has been devoted to the determination of a complete, independent set of invariants and the associated canonical forms for the linear system Σ using various choices of the group $\mathcal{G}$.

Our objective in this report is to extend the basic results obtained for the linear system Σ to the case in which a quadratic cost functional

$$J = \int_t^T [(x,Qx) + 2(x,Su) + (u,Ru)]ds + (x(T),P_0x(T)) \quad ,$$

with $Q = Q'$, $R > 0$, $P_0 = P_0'$, is adjoined. Thus, we deal with the question of determining a complete, independent set of invariants for the so-called "linear-quadratic-gaussian" (LQG) problem. It will be shown that choosing $\mathcal{G}$ to be what we term the "Riccati group", such a set of invariants may be obtained for the LQG problem.

The general plan of the paper is to present, first of all, a brief review of the basic ideas of invariant theory with special emphasis on their use in linear system problems. We then give a formulation of the LQG problem which is particularly suitable for our purposes and define the transformations comprising the "Riccati" group. The next section contains the main results of the paper, namely a complete, independent set of invariants and the associated canonical form for the LQG problem. The paper concludes with a discussion of unresolved issues, as well as the connections of the current results with the generalized X-Y functions introduced in [9-11].

2. Invariant Theory and Linear Systems*

In general terms, invariant theory addresses the following situation: a fixed group $\mathcal{G}$ acts on certain mathematical "quantities" q. An (absolute) algebraic invariant of q with respect

---

* A more complete treatment of these matters is given in [22].