

Boundary Value Problems of Linear Partial Differential Equations for Engineers and Scientists

Shien-siu Shu

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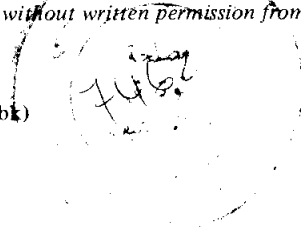
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**BOUNDARY VALUE PROBLEMS OF LINEAR PARTIAL
DIFFERENTIAL EQUATIONS FOR ENGINEERS AND SCIENTISTS**

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To My Family,
Wife Irene,
and Children: Alice, David, Frank, and Jeannie

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PREFACE

A philosopher once said that mathematics is nothing but training; however, engineers have their own motivation to learn mathematics. Because they deal with complicated practical problems and are eager to find useful methods, engineers often face the task of constructing reasonable models, for which familiarity with analytical solutions of simple cases can throw light on the understanding of the more complex situations.

The present text is a revised version of the author's lecture notes in a graduate course of applied mathematics, developed at the Illinois Institute of Technology in the early fifties and expanded at Purdue University. The text is based on the idea that engineering students may find it more interesting to learn mathematics through the introduction of concrete examples. In carrying out this task, I have tried to organize the material in a logical order that transmits the package of mathematical knowledge and methods to the students in an efficient manner. Many problems, utilizing the existing laws of science, can be formulated in mathematical form; often the formulation leads to boundary-value problems in linear partial differential equations, for which solutions are then required. Thus, various standard methods of solution are naturally introduced, employing the techniques of Fourier series, orthogonal functions, Laplace and other transforms, Green's functions, Riemann's conformal mapping, etc. The only previous knowledge assumed as prerequisites is advanced calculus and elementary ordinary differential equations.

In order to enhance the students' enthusiasm for learning the material, exercises are chosen carefully. This enables the students not only to find their own way to solutions, but also strengthens their practical grasp of the subject under discussion in the text. Supplementary knowledge necessary for some treatments -- for instance, the elementary theory of analytic functions -- is included in an appendix. Cauchy's residue theorem is particularly important

in the evaluation of definite integrals of the type that appear by employing operational methods.

The present text is not intended as a complete treatment of boundary-value problems in linear partial differential equations. A conscientious reader is encouraged to consult the additional material listed in the references.

Finally, the author wishes to thank Mrs. Terri Moore for her painstaking fine work in typing the notes in the present form.

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