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ON RECURRENCES IN GENERALIZED ARITHMETIC TRIANGLE

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ABSTRACT. In the present paper, we consider the generalized arithmetic triangle called GAT which is structurally identical to Pascal's triangle for which we keep the Pascal's rule of addition and we replace both legs by two sequences $(a_n)_{n\geq 1}$ and $(b_n)_{n\geq 1}$ with $a_0 = b_0 = \Omega$. Our goal is to describe the recurrence relation associated to the sum of elements lying along a finite ray in this triangle. As consequences, we obtain some combinatorial properties and we establish that the sum of elements lying along a main rising diagonal is a convolution of generalized Fibonacci sequence and another sequence which one will determine. We also precise the corresponding generating function. Further, we establish some nice identities by using the Morgan-Voyce phenomenon. Finally, we generalize the Golden ratio.

1. Introduction

Pascal's triangle appears in numerous situations and has many intriguing properties. Over the years, a number of Pascal-like triangular arrays have been developed. Koshy [20, 21] and Belbachir and Szalay [7] collected various generalizations. The generalization of Ensley [13] called *Generalized Arithmetic Triangle,* the Ensley's GAT for short, changed the legs called generator sequences to an arbitrary sequences of real numbers $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. Then the GAT depends on the generator sequences $(a_n)_{n\geq 0}$ and $(b_n)_{n\geq 0}$. Ensley [13] studied the Fibonacci triangle, the case where $a_n = b_n = F_n$, the Fibonacci numbers. The cases, where $(a_n, b_n) = (0,\ F_n)$, $(a_n, b_n) = \left(0, \frac{1}{n}\right)$ and $(a_n, b_n) = (F_{2n-1}, F_{n-1})$, were treated by Dil and Mezo [12]. In [8], we can also find an interesting description of Fibonacci and Lucas triangles. As Fibonacci numbers can be generated from the rising diagonals of Pascal's triangle, Feinberg, see [14], was motivated to develop a similar triangle that would generate Lucas numbers from its rising diagonal. This triangle is Lucas triangle which is the particular case of Ensley's GAT when $a_n = 1$ and $b_n = 2$. For other recent works on the subject, we refer to references [2, 3]

In the present paper, we consider similar construction of Ensley's GAT and our aim is to determine the recurrence relation associated to the sum of elements lying along a finite ray. Then, we establish the corresponding generating function. Also, we study the Morgan-Voyce phenomenon that leads us to establish new identities. Finally, we provide some interesting applications, among

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